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[Flux, Sir Alfred William]

SOLUTIONS OF EXAMPLES

IN

ELEMENTARY HYDROSTATICS

BY

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PREFACE.

I HAVE been frequently asked to produce solutions of the examples in my Treatise on Elementary Hydrostatics, but the pressure of other work has prevented me from undertaking the task of preparing them.

These solutions have been almost entirely drawn up by Mr A. W. FLUX, Fellow of St John's College, and I am much indebted to him for the labour which he has bestowed upon the work.

I hope that they will be found to be useful and helpful, both to teachers and to students.

No figures have been given, but the student will find no serious difficulty in drawing figures for himself when necessary, and he will find it greatly to his advantage to do so.

W H. BESANT.

January 1891.

INTRODUCTORY NOTE.

IN these solutions the expression employed for the weight of a volume V of fluid is sometimes wV and sometimes $g\rho V$.

It must be borne in mind that when wV is used, w is the intrinsic weight, that is the weight of unit volume of the fluid, expressed in pounds weight, so that wV is the number of pounds weight of the volume V of fluid. When $g\rho V$ is employed the unit of force is the poundal, that is, the force required to produce the unit of acceleration in the unit of mass.

If we take one foot, one second, and one pound as the units, the poundal is roughly equal to the weight of half an ounce.

It must be clearly seen that the poundal is an absolute unit of force, independent of time and place, whereas the weight of a pound, that is, the attraction of the Earth upon a pound, combined with the effect of the Earth's rotation, is both local and temporary;

it is different at different places, and it changes at the same place from time to time.

The Earth is not homogeneous, nor is it a sphere, and it is not at rest, but is in a state of rotation about its axis; hence it is that weight depends upon locality.

Further, changes are perpetually taking place in portions of the masses which constitute the Earth, both upon its surface, and beneath its surface; hence it is that the weight of a body at the same place is not an absolutely fixed quantity.

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CHAPTER I.

EXAMINATION.

4\. (1) THE unit of area is a square inch, the pressure on which is $10\frac{1}{2}$ pounds.

(2) The unit of area is four square inches, the pressure on which is 42 pounds.

5\. Suppose h so small that the pressure may be considered uniform over the rectangle, the area of which is bh.

This uniform pressure is

$$wbh\,(a+h)/bh = w\,(a+h) = wa$$

when h vanishes, i.e. at any point in the upper side.

6\. Area of larger pipe $= 9 \times$ area of smaller pipe;

$\therefore$ required force $= 9 \times 20 = 180$ lbs.

8\. Area of $B = 64 \times 36 \times 36 \times$ area of A;

$\therefore$ weight supported by $B = 64 \times 36 \times 36$ lbs. $= 37\frac{1}{35}$ tons.

9. If a be the length of the side of the square, $2b$ the other side of the rectangular lid, W its weight, p the pressure when the lid is on the point of lifting,

Taking moments about the line of the hinge

$$pa^2 \times a = W \times b \text{ or } p = \frac{Wb}{a^3}.$$

11. Let pq be parallel to BC at a small distance h from PQ.

Then pressure on

$$PpqQ = p\,[(x+h)^2 - x^2] = p\,(2xh + h^2).$$

And area of $PpqQ = PQ \times h$ very nearly.

$\therefore$ the pressure at any point in PQ is

$$\frac{2xp}{PQ} = \frac{2p\,.\,d}{BC},$$

if d be the perpendicular from A on BC, and the pressure is uniform over the triangle.

12. The pressure is $\dfrac{10000}{14\frac{1}{2} \times 36\pi}$ atmospheres.

$\therefore$ the compression $= \dfrac{2 \times 10000}{29 \times 36\pi} \times {\cdot}00005$ of the whole volume;

$\therefore$ distance through which piston is compressed

$$= \frac{10 \times 12}{29 \times 36\pi} = \frac{1}{27{\cdot}3} \text{ inch.}$$

CHAPTER II.

EXAMINATION.

2. WEIGHT of a cubic foot of mercury $= 13{\cdot}568 \times 1000$ ounces $= 848$ lbs.

3. The required weight

$$= 27 \times 1728 \times 5 \times {\cdot}45 = 104976 \text{ lbs.}$$

4. Let V, s be the volume and specific gravity of one fluid, mV, ns of the other, σ the specific gravity of the mixture.

$$\sigma(m+1)V = Vs + mnVs;$$

$$\therefore\ s = \sigma\frac{1+m}{1+mn}.$$

5. $2V$, V being the volumes of the two fluids which are mixed, $2V$ is the volume of the mixture.

$$\therefore\quad \text{density of mixture} = \frac{2V\rho + 2V\rho}{2V} = 2\rho.$$

6. The required weight

$$= 27 \times {\cdot}12 \times 1000 \text{ ozs.} = 202\tfrac{1}{2} \text{ lbs.}$$

7. A cubic inch of water weighs $\frac{1000}{1728 \times 16}$ lbs.

$\therefore$ the sp. gr. of the substance

$$= \frac{1625}{3456} \times \frac{16 \times 1728}{1000} = 13.$$

8. s, s', s'', σ being the densities of the three fluids and of the mixture,

$$s+s'+s''=3\sigma; \qquad \therefore\ s=3\sigma-s'-s''.$$

9. Let the resulting volume be $V+V'-v$.

Then $$s(V+V'-v)=V\sigma+V'\sigma';$$

$$\therefore\ v=\frac{V(s-\sigma)+V'(s'-\sigma')}{s}.$$

10. Let $2V$ be the volume of each fluid mixed.

The volume of the mixture is $3V$. Its sp. gr. being σ,

$$3V\sigma=2Vs+4Vs; \qquad \therefore\ \sigma=2s.$$

Examples.

1. The densities of the n fluids being

$$\rho,\ 2\rho,\ 3\rho, \ldots\ldots n\rho$$

when equal volumes are mixed, the density of the mixture is

$$\frac{1+2+3+\ldots+n}{n}\rho=\frac{n+1}{2}\rho.$$

When the volumes are in the ratios of $1.2.3\ldots n$, the density of the mixture

$$=\frac{1^2+2^2+\ldots+n^2}{1+2+\ldots+n}\rho=\frac{2n+1}{3}\rho.$$

When the volumes are in the ratios of

$$n.\overline{n-1}\ldots3.2.1,$$

the density of the mixture

$$=\frac{n+2(n-1)+3(n-2)+\ldots+n}{n+\overline{n-1}+\ldots+2+1}\rho=\frac{n+2}{3}\rho.$$

2. s_1, s_2 being the sp. gr. of the two fluids,

$$\sigma=\tfrac{1}{2}(s_1+s_2), \qquad \sigma'=\tfrac{1}{3}(s_1+2s_2);$$

$$\therefore \quad s_1=4\sigma-3\sigma', \qquad s_2=3\sigma'-2\sigma.$$

3. In the new units $g=\frac{32}{3}$.

$\therefore$ weight of a cubic yard of standard $=\frac{32}{3}\times$ unit of weight $=\dfrac{32}{3}\times 25$ ozs. $=\dfrac{32\times 25}{3\times 27\times 1000}\times$ weight of 1 cubic yard of water.

$\therefore$ density of standard : density of water $=4:405$. Or we may conduct the argument as follows; let ρ be the density of water referred to the standard substance; then, taking a cubic yard of water,

$$V=1, \text{ and } W=\tfrac{27000}{25},$$

and the equation, $W=g\rho V$, becomes

$$\tfrac{27000}{25}=\tfrac{32}{3}\rho, \text{ and } \therefore \rho=\tfrac{405}{4}.$$

4. If ρ is density of water referred to standard, the equation, $W=g\rho V$, becomes for the unit of volume of water,

$$\frac{64000}{16\times 3\times 112}=\frac{32}{4}\rho, \text{ and } \therefore \rho=\frac{125}{84}.$$

5. In $W=g\rho V$, a cubic foot of standard weighs 32 lbs., and in $W=sV$, it weighs 1 lb.

6. If when V, s, and ρ are each unity, W is unity in each equation, it follows that $g=l$, and therefore, if t seconds is the unit of time,

$$32t^2=1.$$

7. In the new units $g=\frac{32}{3}\times\frac{1}{4}=\frac{8}{3}$.

In $W=g\rho V$ weight of 1 cubic yard of the standard

$$=\tfrac{8}{3} \text{ lbs.}$$

In $W=sV$ weight of 1 cubic yard of the standard $=1$ lb.

8. Let l feet be the unit of length in $W=g\rho V$.

Then $$g=\frac{32\times 8}{l}.$$

$\therefore$ weight of l^3 cub. feet of the standard $=\frac{32\times 8}{l}$ units of weight.

But, from $W=sV$, weight of unit volume of standard is unit of weight,

$$\therefore\ l^3=\frac{256}{l},\text{ or } l=4.$$

9. In $W=g\rho V$, weight of 27 cubic yards of the standard $=\frac{32\times 16}{9}$ units of weight.

In $W=sV$, weight of 27 cubic yards of the standard $=1$ unit of weight.

10. Let s_1, s_2 be the specific gravities of the fluids,

$$\frac{s_1+s_2}{2}=\frac{4}{3}\cdot\frac{2}{\frac{1}{s_1}+\frac{1}{s_2}};$$

$$\therefore\ 3s_1^2-10s_1s_2+3s_2^2=0;$$

$$\therefore\ s_1:s_2=1:3 \text{ or } 3:1.$$

11. Let x gallons be the quantity required,

$$\frac{x+10\cdot31}{x+10}=1\cdot021;\ \therefore\ x=\frac{1}{\cdot21}=4\cdot7619\ldots$$

12. Volume of earth

$$=\tfrac{4}{3}\pi\,(1\cdot275)^3.\,10^{27}\text{ cubic centimetres.}$$

$\therefore$ mass of earth $=\frac{4}{3}\pi\,(1\cdot275)^3\times 5\cdot67\times 10^{27}$ grammes
$=6\cdot15\times 10^{27}$ grammes about.

CHAPTER III.

EXAMINATION.

2. (1) THE pressure on a unit of area (a square inch) = the weight of $\frac{100}{144}$ cubic feet of water $= 43\frac{29}{72}$ lbs.

(2) Pressure is now increased by the atmospheric pressure, i.e. by about $14\frac{1}{2}$ lbs. per square inch, and is therefore about 58 lbs.

4. Neglecting atmospheric pressure, the pressure on a square inch = the weight of $\frac{170}{144}$ cubic feet of water $= 73\frac{113}{144}$ lbs.

6. The depth of the centre of gravity of the triangle is $\dfrac{1}{2\sqrt{3}}$ feet.

Its area is $\sqrt{3}/4$ square feet.

$\therefore$ the pressure on it = the weight of $\frac{1}{8}$ cubic foot of water $= 125$ ozs.

8. If a cylinder with vertical generating lines be described on the same base as the cone, the pressure on the curved surface of the cone is the weight of the liquid which would fill the space between the cone and cylinder, i.e. twice the weight of the liquid in the cone.

9. If we suppose the density of the area to vary as the depth below the surface, its centre of gravity would be the centre of pressure. Now this centre of gravity is evidently at a greater depth than that of a uniform area with the same boundary, unless the area be horizontal, when they coincide.

11. Let h be the length of the vertical side of the rectangle, x the depth of the required line, b the breadth of the rectangle.

The whole pressure on the rectangle $= w \cdot \frac{h}{2} \cdot hb.$

The pressure on the upper portion $= w \cdot \frac{x}{2} \cdot xb.$

$$\therefore x^2 = \tfrac{1}{2}h^2 \quad \text{or} \quad x = h/\sqrt{2}.$$

12. If x_r be the depth of the rth line of division, we have as in (11),

$$x_r^2 = \frac{r}{n} h^2;$$

$$\therefore \ x_r = \frac{r}{\sqrt{n}} h.$$

13. h being the height, $2a$ the base of the triangle, x the depth of the dividing line, whose length $\therefore = 2 \cdot \frac{xa}{h}$.
Whole pressure on triangle $= w \cdot \tfrac{2}{3}h \cdot ha.$

Pressure on upper portion $= w \cdot \tfrac{2}{3}x \cdot x\frac{xa}{h};$

$$\therefore \ x^3/h = \tfrac{1}{2}h^2;$$

$$\therefore \ x = h/\sqrt[3]{2}.$$

Examples.

1. Let A be the sectional area of the cylinders, W the weight of the piston, h, k the heights of the water in the open and closed cylinders. The position of equilibrium is given by

$$w \cdot hA = wkA + W,$$

or

$$h - k = W/wA.$$

The volume of water $(h+k)\,A$ in the two cylinders being known, h and k are at once determined.

2. The weight must be equal to the weight of $8 \times (2\frac{1}{2})^2$ cubic feet of water, i.e. 3125 lbs.

3. Let h be the height, b the breadth of the rectangle, and let the line cut the lower side at a distance $b-x$ from the opposite corner.

Pressure on whole rectangle $= w \cdot \frac{h}{2} \cdot hb$.

Pressure on triangular part $= w \cdot \frac{2h}{3} \cdot \frac{hx}{2}$.

$$\therefore \quad \frac{2x}{3} = \tfrac{1}{2}b \quad \text{or} \quad x = \tfrac{3}{4}b.$$

4. Let h be the whole length of tube occupied by the two liquids, x the height of their common surface above B.

The height of the surface of the lighter liquid above the common surface $= h/2\sqrt{2}$.

That of the surface of the heavier liquid is

$$\left(\frac{h}{2} - 2x\sqrt{2}\right)\Big/\sqrt{2};$$

$$\therefore \; 2\left(\frac{h}{2} - 2x\sqrt{2}\right) = \frac{h}{2} \quad \text{or} \quad x = h/8\sqrt{2}.$$

5. Area of curved surface $= \pi$ square feet.

Pressure on it due to weight of water $=$ weight of $\frac{\pi}{2}$ cubic feet of water.

Pressure on it due to weight of piston $= \pi \times 4 \div \frac{\pi}{4} = 16$ lbs.

$\therefore$ whole pressure $= 16 + \frac{125}{4}\pi$ lbs.

6. The cylinder being full of water, the effect is simply that which would be produced by increasing the weight of the piston by 1 lb., i.e. the whole pressure is increased by 4 lbs.

The pressure at a depth h is $\frac{20}{\pi} + \frac{125}{2}h$ lbs. per square foot.

7. (1) When the vessel is full, the water will overflow as the lead is immersed, the pressure on the base being unchanged.

(2) If the vessel is not full, the pressure on the base is increased by the weight of a quantity of water equal in volume to the piece of lead.

8. Let h be the height, r the radius of the cylinder.

Pressure on curved surface $= wrh \cdot 2\pi r$.

Pressure on each end $= wr \cdot \pi r^2$.

Weight of fluid $= wh \cdot \pi r^2$.

$$\therefore \quad 2hr + 2r^2 = 3hr,$$

or

$$2r = h.$$

9. Let the vertical through C meet AB in D, and let θ be the angle made by CA, CB with the surface.

The depth of A is $b \sin\theta$, that of B is $a \sin\theta$;

$$\therefore \quad \text{the depth of } D = \frac{2ab \sin\theta}{a+b}.$$

For CD bisects ACB and therefore $AD : DB = b : a$.

The depth of the C.G. of ACD

$$= \tfrac{2}{3}\left(b \sin\theta + \frac{2ab \sin\theta}{a+b}\right)\Big/2 = \frac{\sin\theta}{3} \cdot \frac{b(b+3a)}{a+b}.$$

The depth of the C.G. of $BCD = \dfrac{\sin\theta}{3} \cdot \dfrac{a(a+3b)}{a+b}$.

The areas of ACD, BCD are as $AD : BD = b : a$;

$\therefore$ the pressures on them are as $b^2(b+3a) : a^2(a+3b)$.

10. Let ABC be the triangle, AB being in the surface. If O be any point in the line CD, joining C to the middle point of AB,

The pressures on the triangles CAD, CBD are equal and the pressures on OAD, OBD are equal.

$\therefore$ the pressures on COA, COB are equal.

If $OD=x$ and $CD=h$,

Pressure on OAB : pressure on $CAB=x^2 : h^2$.

$\therefore$ if O be the point required,

$$x^2=\tfrac{1}{3}h^2 \quad \text{or} \quad x=h/\sqrt{3}.$$

11. Let $\qquad AD=x.$

Then pressure on ABD : pressure on $ABC=x^2 : b^2$;

$$\therefore \quad x^2=\tfrac{1}{2}b^2.$$

And $\qquad AD : DC=x : b-x=1 : \sqrt{2}-1.$

12. Let x inches be the side of the square, $2w$, $3w$ the intrinsic weights of the two fluids.

The pressure on the part in the lighter fluid

$$=2w \,.\, 2 \,.\, 4x=16wx.$$

That on the lower part $=\left(2w \,.\, 4+3w \,.\, \frac{x-4}{2}\right) x\,(x-4).$

These being equal, we obtain

$$x=\tfrac{4}{3}\,(1 \pm \sqrt{10}).$$

The upper sign must be taken, since x is essentially positive.

13. Let p be the perimeter of the cylinder, h the depth of each fluid.

The fluid pressures on the three portions are as

$$\rho \,.\, \frac{h}{2} \,.\, ph \;:\; \left(\rho h+2\rho \,.\, \frac{h}{2}\right) ph \;:\; \left(\rho h+2\rho h+3\rho\,\frac{h}{2}\right) ph;$$

$\therefore$ these pressures are as $1 : 4 : 9$.

14. Let α, $\pi-\alpha$, be the angles subtended at the centre of the tube by the two portions of fluid, ρ, ρ' their densities, a the radius of the circular tube, θ the inclination to the vertical of the bounding diameter.

The two expressions for the pressure at the common surface are in the ratio of

$$\rho\,[a\cos\theta - a\cos(\theta+a)],$$

and $$\rho'\,[-a\cos(\theta+a) - a\cos\theta].$$

Equating these we find

$$\tan\theta = \cot a + \frac{1}{\sin a}\,\frac{\rho'+\rho}{\rho'-\rho}.$$

15. Let h be the length of the cylinder, A its sectional area, θ, $\frac{\pi}{2}-\theta$, the inclinations of the axis to the vertical, w the intrinsic weight of the fluid.

$$P = wA\,.\,h\cos\theta,$$

$$P' = wA\,.\,h\sin\theta\,;$$

$$\therefore\quad [P^2+P'^2]^{\frac{1}{2}} = whA = \text{weight of fluid displaced.}$$

16. Let ρ be the density of the fluid, r the radius of the cylinder, h the distance of the centre of the sphere from the base of the cylinder.

$$\tfrac{2}{3}\pi r^3 + \pi r^2\,.\,2r = \pi r^2\,.\,h\,;\quad\therefore\quad h = \tfrac{8}{3}r.$$

Measuring in poundals,

pressure on curved surface of cylinder before the introduction of the sphere $= g\rho\,.\,r\,.\,2\pi r\,.\,2r = 4g\rho\pi r^3$.

After the sphere is introduced it is

$$\left(\tfrac{7}{8}\,.\,\frac{4g\rho\,.\,\frac{4}{3}\pi r^3}{\pi r^2} + g\rho\,.\,\frac{h+r}{2}\right)2\pi r\,.\,h = 32\,g\rho\pi r^3.$$

The increase $= 28g\rho\pi r^3 = 14 \times$ weight of fluid in poundals.

17. Let ρ, ρ', ρ'' be the three densities,

x, y the depths of the common surfaces.

Then $$\rho x = \rho'' y - \rho'(y-x),$$

or $$(\rho-\rho')\,x = (\rho''-\rho')\,y\,;$$

$\therefore$ $\rho-\rho'$ and $\rho''-\rho'$ are of the same sign.

And if $y > x$, $\rho''-\rho' < \rho-\rho'$;

$\therefore$ the order of magnitude is $\rho\,.\,\rho''$, ρ'.

And these being in A.P.,

$$\rho-\rho'=2(\rho''-\rho');$$
$$\therefore\ y=2x.$$

18. Let x be the required distance.

Then
$$\rho' a\left(1-\frac{1}{\sqrt{2}}\right)=\rho(a-x),$$
$$\therefore\ x=a\left[1-\frac{\rho'}{\rho}\left(1-\frac{1}{\sqrt{2}}\right)\right].$$

19. Referring to Examination (9), the area whose C.G. is the centre of pressure becomes more nearly of uniform density as the plane area is lowered. Hence its C.G. approaches and ultimately coincides with that of the area of uniform density.

20. Produce AB, DC to meet in E.

Let $DE=h\alpha$ and $\therefore\ CE=h\beta$.

Pressure on $ECB=w\,.\,\frac{\beta}{3}\,.\,\frac{\beta^2 h}{2}$, and depth of centre of pressure $=\frac{\beta}{2}$.

Pressure on $EDA=w\,.\,\frac{\alpha}{3}\,.\,\frac{\alpha^2 h}{2}$, and depth of centre of pressure $=\frac{\alpha}{2}$.

$\therefore$ depth of centre of pressure of $ABCD$

$$=\frac{\alpha^3\,.\,\frac{\alpha}{2}-\beta^3\,.\,\frac{\beta}{2}}{\alpha^3-\beta^3}=\tfrac{1}{2}\left(\frac{\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3}{\alpha^2+\alpha\beta+\beta^2}\right).$$

21. Let x be the height of the lower portion, h that of the triangle.

The areas are as $x^2 : h^2$.

The depth of the C.G. of the upper portion is

$$\frac{h^2\,.\,\frac{h}{3}-x^2\left[h-x+\frac{x}{3}\right]}{h^2-x^2}=\tfrac{1}{3}\,.\,\frac{h^3-3hx^2+2x^3}{h^2-x^2}.$$

The pressure on it is $\frac{1}{3} g\rho (h^3 - 3hx^2 + 2x^3)$.

That on the lower portion is

$$g\left[\rho (h-x) + 14\rho \frac{x}{3}\right] x^2 = \tfrac{1}{3} g\rho x^2 (3h + 11x).$$

Equating these we obtain

$$(h-3x)(h^2+3hx+3x^2)=0.$$

The real root $x = \frac{h}{3}$ gives the solution of the present problem.

$\therefore$ the areas of the two portions are as

$$h^2 - x^2 : x^2 = 8 : 1.$$

22. Let $2h$ be the height of the floodgate, b its width, P, p the pressures at the upper and lower corners.

$$2(P+p) = g\rho h \,.\, 2hb - g\rho \,.\, \frac{h}{2} \,.\, hb = \tfrac{3}{2} g\rho h^2 b,$$

$$2p \,.\, 2h = g\rho h \,.\, 2hb \,.\, \frac{4h}{3} - g\rho \frac{h}{2} \,.\, hb \,.\, \frac{2h}{3} = \tfrac{7}{3} g\rho h^3 b;$$

$$\therefore \quad p = \tfrac{7}{12} g\rho h^2 b.$$

$$P = \tfrac{1}{6} g\rho h^2 b.$$

23. Let $\rho \,.\, \rho'$ be the densities, h the depth of each liquid.

Then $$3\rho \,.\, \frac{h}{2} = \rho h + \rho' \frac{h}{2};$$

$$\therefore \quad \rho = \rho'.$$

24. Let l feet be the unit of length.

The new value of g is $\dfrac{32}{l}$;

$$\therefore \quad p = \frac{32}{l} \rho z \text{ lbs.}; \quad \therefore \quad \frac{32}{l} \text{ lbs.} = 1 \text{ unit of weight} = 16 \text{ lbs.};$$

$$\therefore \quad l = 2.$$

25. Let l feet, t seconds be the required units.

l feet per t seconds is the unit of velocity;

$$\therefore \quad l=t.$$

$g=\dfrac{32t^2}{l}$ in the new units $=32l$,

when $\rho=1$, $z=1$, $p=32l$ ounces.

And this pressure is the weight of l^3 cubic feet of water, i.e. 1000 l^3 ounces;

$$\therefore \quad 1000l^3=32l,$$

$$l=\frac{2}{5\sqrt{5}}.$$

26. Let t seconds be the unit of time.

$g=\dfrac{32t^2}{3}$ in the new units.

Putting ρ and z unity, p=weight of unit volume of the standard$=1000$ lbs.;

$$\therefore \quad 1000=\frac{32t^2}{3},$$

$$\text{or } t=\frac{5\sqrt{15}}{2}.$$

27. Area of surface of sphere$=4\pi\times(\frac{1}{2})^2=\pi$ square feet;

$\therefore$ pressure on surface=weight of $\dfrac{3\pi}{2}$ cubic feet of water.

The new value of g is $32\times(\frac{1}{4})^2=2$;

$\therefore$ the numerical value of the pressure$=3\pi$.

28. Let a, b, c be the breadths of the sides of the prism, δ, ϵ, ζ the lengths of the edges.

P is proportional to $a\,\dfrac{\epsilon+\zeta}{2}$, i.e. to $\sin a\,\dfrac{\epsilon+\zeta}{2}$,

Q to $\sin\beta\,.\,\dfrac{\zeta+\delta}{2}$, R to $\sin\gamma\,\dfrac{\delta+\epsilon}{2}$.

$\therefore$ P cosec $a+Q$ cosec $\beta+R$ cosec γ is proportional to $\delta+\epsilon+\zeta$, which being three times the depth of the C.G. of the prism remains unchanged.

29. Let a be the length of the edge of the cube,
h the depth of the fluid in it.

The width of the surface of the fluid $=a-h$;

$\therefore$ the volume of the fluid $=ah.\dfrac{a+\overline{a-h}}{2}=\frac{1}{2}ah\,(2a-h)$.

Since this $=\frac{1}{4}a^3$, $h=a\left(1-\dfrac{1}{\sqrt{2}}\right)$.

The pressure on the loose face is $g\rho.\dfrac{h}{2}.ah\sqrt{2}$.

Its moment about the hinge $=g\rho\dfrac{ah^2}{\sqrt{2}}.\dfrac{h\sqrt{2}}{3}$.

The moment of the weight of the face $=w.\dfrac{a}{2\sqrt{2}}$.

Equating these and inserting the value of h,

$$\frac{w}{\frac{1}{4}g\rho a^3}=\tfrac{4}{3}(\sqrt{2}-1)^3=\tfrac{4}{3}(5\sqrt{2}-7).$$

30. Let the box be tilted through an angle θ.

(i) About the edge on the same side as the hinge.

Pressure on lid $=g\rho.\dfrac{a\sin\theta}{2}.a^2$.

Its moment about the hinge $=g\rho.\dfrac{a^3\sin\theta}{2}.\dfrac{2a}{3}$.

Moment of weight of lid about hinge $=w.\dfrac{a\cos\theta}{2}$;

$\therefore$ when $\tan\theta=\frac{3}{2}\dfrac{w}{g\rho a^3}$ the water begins to escape.

(ii) About the edge diagonally opposite to the hinge.

Moment of fluid pressure about hinge $=g\rho.\dfrac{a^3\sin\theta}{2}.\dfrac{a}{3}$;

$\therefore$ when $\tan\theta=3.\dfrac{w}{g\rho a^3}$ the water begins to escape.

(iii) About either of the other edges.

Moment of fluid pressure about hinge $=g\rho.\dfrac{a^3\sin\theta}{2}.\dfrac{a}{2}$.

Moment of weight about hinge $= w \cos\theta \cdot \dfrac{a}{2}$.

$\therefore$ When $\tan\theta = 2\dfrac{w}{g\rho a^3}$ the water begins to escape.

These values of $\tan\theta$ are as 3 : 6 : 4.

31. The volume of the wine $= \frac{2}{3} \times \frac{12}{11} = \frac{8}{11}$ that of the water.

The density of the mixture $= \dfrac{8 \cdot 11 + 11 \cdot 12}{8 + 11} = \dfrac{220}{19}$.

Taking the density of water as 12.

The depth of pure wine $= \frac{4}{19}$ of the cylinder.

That of the mixture $= \frac{1}{2}$ of the cylinder, and of water, $\frac{11}{38}$ of the cylinder.

The pressure on the curved surface in contact with the water is proportional to

$$\left(11 \times \tfrac{4}{19} + \tfrac{220}{19} \cdot \tfrac{1}{2} + 12 \cdot \tfrac{11}{76}\right) \tfrac{11}{38}.$$

That on the rest of the surface is proportional to

$$11 \cdot \tfrac{2}{19} \cdot \tfrac{4}{19} + \left(11 \cdot \tfrac{4}{19} + \tfrac{220}{19} \cdot \tfrac{1}{4}\right) \tfrac{1}{2}.$$

And these two quantities are equal.

32. Let A be the area of the fluid surface, h the depth of the vertex below that surface in the first case.

The area of surface in contact with fluid is $\dfrac{A \cos\theta}{\sin\alpha}$.

$\therefore$ Fluid pressure on it $= g\rho \cdot \dfrac{h}{2} \cdot \dfrac{A \cos\theta}{\sin\alpha}$.

Let A' be the area of the fluid surface, k the depth of the fluid when the cone is vertical.

The fluid pressure $= g\rho \cdot \dfrac{k}{2} \cdot \dfrac{A'}{\sin\alpha}$.

But $A' \cdot k = A \cdot h$ since the volume of fluid is the same.

$\therefore$ The pressure is changed in the ratio $\cos\theta : 1$.

33. The resultant pressure on the curved surface is due to the weight of the fluid and the upward pressure of the base, which are equal and opposite parallel forces of magnitude W, and the distance between their lines of action is $\frac{1}{2}h \tan\alpha$.

CHAPTER IV.

EXAMINATION.

5. THE c. g. of the two weights must be at the middle of the plank.

7. One third of the cylinder must be immersed;

$$\therefore \text{ its length is 12 feet.}$$

8. The specific gravities of the fluids are respectively $\frac{4}{5}$ and $\frac{3}{4}$ that of the solid, and are $\therefore$ as 16 : 15.

9. In each case the force is equal to the weight of the water displaced by six inches of the cylinder.

10. One-half the perimeter of the triangle must be immersed, i.e. 3/4 of each of the two lower sides.

12. One-half the sphere is immersed, equilibrium being established with the nail at the highest point (unstable) or at the lowest point (stable).

13. If x cubic feet be the required volume:

Volume of water displaced $=4+x$ cubic feet.

$$\therefore\ 4+x=4\,.\,\tfrac{1}{2}+x\,.\,\tfrac{7}{2}.$$

$$\therefore\ x=\tfrac{4}{5}.$$

14. The pressure is that of the wood and superincumbent volume of water, which is of equal volume with the wood.

15. The water will flow from that vessel in which the pressure at the base is the greater.

16. Let V, V', be the volumes, σ, σ' their specific gravities. ρ the sp. gr. of water.

The weights of the two bodies in water being equal

$$V(\sigma-\rho)=V'(\sigma'-\rho),$$

$$\text{or } \frac{V}{V'}=\frac{\sigma'-\rho}{\sigma-\rho}.$$

Examples.

1. The volume immersed is equal to that not immersed, and the c. g. of the part immersed coincides with that of the fluid displaced, so that the c. g.'s of the two halves are in a vertical line. Hence, if inverted, the solid will float in equilibrium.

2. Let the volume of the granite be x cubic yards.

$$\cdot918\,(1-x)+2\cdot65x=\tfrac{23}{25}=\cdot92.$$

$$\therefore\ x=\tfrac{1}{866}.$$

3. Half the area of the triangle must be immersed.

$\therefore$ The altitude is divided by the surface of the liquid in the ratio $\qquad 1 : \sqrt{2}-1.$

4. Half the volume of the cone being immersed in each case, the portions of the axis immersed are in the ratio $\qquad 1 : \sqrt[3]{2}-1.$

5. Let w be the weight, V the volume of the body.

Then $\qquad w=w_1+Vs_1=w_2+Vs_2=w_3+Vs_3.$

$$\therefore\ w_1(s_2-s_3)+w_2(s_3-s_1)+w_3(s_1-s_2)=0.$$

6. $\frac{1}{4}$ of AB is above the surface and $\therefore$ $\frac{1}{8}$ of the triangle.

If its density be ρ, and that of the liquid unity, taking moments about A of the weight of the lamina and of the displaced fluid

$$\rho = 1 - \tfrac{1}{8} \cdot \tfrac{1}{2} = \tfrac{15}{16}.$$

7. Let h be the depth of the upper liquid, x the length of the cylinder in the lower liquid, $\therefore$ $2h - x$ that in the upper.

$$\frac{7\rho}{4} \cdot 2h = (2h - x)\,\rho + x \cdot 2\rho,$$

giving $x = \frac{3}{2}h$, i.e. one quarter of the cylinder is in the upper liquid.

8. The c. g. of the weighted rod is one-third of its length from the weighted end.

$\therefore$ The c. g. of the displaced liquid is at this point.

$\therefore$ Two-thirds of the length of the rod is immersed and the density of the liquid $= \frac{3}{2}$ density of weighted rod $= \frac{9}{4}$ that of the wood.

9. Let the portion not immersed be to the whole as $x : 1$, w its weight, s the specific gravity.

The whole weight is $\dfrac{w}{x}$, the whole volume $\dfrac{w}{xs}$.

The weight of water displaced $= \dfrac{w\,(1 - x)}{xs}$.

$$\therefore\ s = 1 - x \text{ or } s + x = 1.$$

Now sx is to be as great as possible, $\therefore$ $s = x = \frac{1}{2}$.

10. Weight of water to be displaced

$$= \pi \cdot \frac{9}{4} \cdot \frac{9}{2} \cdot \frac{1000}{1728} = \frac{375\pi}{64} \text{ ozs.}$$

$\therefore$ The required weight $= \left(\dfrac{375\pi}{64} - 8\right)$ ozs.

11. Evidently one-half that in the previous example.

12. The c. g. of the weighted rod, and $\therefore$ of the displaced fluid, is one-quarter the length of the rod from the weighted end.

$\therefore$ One-half the rod is immersed.

$\therefore$ The weight of fluid displaced by half the rod is double the weight of the rod.

$\therefore$ The density of the fluid is four times that of the rod.

13. The tension of the string = weight of rod − weight of water displaced, and is therefore not changed when the inclination is changed.

14. Let r, r', be the internal and external radii, σ, ρ the densities of the shell and of water.

$$\tfrac{4}{3}\pi\,(r'^3 - r^3)\,\sigma = \tfrac{2}{3}\pi r'^3\sigma.$$

$$\therefore\ \sigma : \rho = r'^3 : 2\,(r'^3 - r^3).$$

15. Let w, w', be the weights of the cone and of the displaced fluid. $w - w'$ is the magnitude of the required force.

If x be the distance of its line of action from the c. g. of the solid cone,

$$(w - w')\,x = w\left(\frac{3}{4}h - \frac{2}{3}h\right) = w\,\frac{h}{12}.$$

$$\therefore\ x = \frac{w}{w - w'}\,\frac{h}{12}.$$

16. $\frac{19}{27}$ of the volume of the cone being immersed, $\frac{8}{27}$ of it is above the surface, i.e. $\frac{2}{3}$ of the axis is above the surface.

17. Let σ, ρ, be the intrinsic weights of the lamina and fluid, $2A$ the area of the lamina, w the weight attached.

BD being the perpendicular from B on AC, let $AC = d$, $AD = x$.

$$w + 2A\sigma = A\rho.$$

The distance of A from the vertical through the c.g. of the half immersed is

$$\frac{2}{3}\left[\frac{x+d}{2}\right].$$

$$\therefore\ A\rho\,.\,\frac{x+d}{3}=wx+2A\sigma\,.\,\frac{d}{2}.$$

$$\therefore\ \rho=3\sigma.$$

18. Let σ, ρ, be the densities of paraboloid and liquid, h the height of the paraboloid, x that of the portion immersed. Area of base of paraboloid varies as the height.

$$\therefore\ \rho x^2=\sigma h^2$$

or $$x=h\sqrt{\frac{\sigma}{\rho}}.$$

19. Let W be the weight of ship and cargo, in tons.

V the volume immersed, in cubic feet.

A the area of the water line section, in square feet.

$$1{\cdot}025\,.\,V=W\,.\,\frac{2240\times 16}{1000}=V+A\,.\,\frac{1}{6}.$$

$$A\,.\,\frac{3}{24}=40\,.\,\frac{2240\times 16}{1000}.$$

Eliminating A and V we obtain $W=2186\frac{2}{3}$.

20. Let $2k$ be the height of the required cylinder.

Half of it being immersed,

$$\pi(r^2-r'^2)\,k=\tfrac{3}{4}\pi r^2\,.\,h-\pi r^2(h-k).$$

$$\therefore\ 2k=\frac{r^2}{r'^2}\,.\,\frac{h}{2}.$$

21. The pressure on the lower half of the curved surface is vertical, and bears to the weight of the water in the cylinder the ratio,

$$\frac{\pi r^2}{2}+2r^2 : \pi r^2=\frac{1}{2}+\frac{2}{\pi} : 1,$$

since it is in equilibrium with the weight of the water in the lower half and the pressures on the horizontal plane separating the two halves. The latter pressure is that due to a depth r of water all over the surface of separation. When the cylinder is vertical, the pressure on one-half the curved surface is horizontal and balances the pressure on the diametral plane. It acts ∴ at a point in the axis $\frac{2}{3}$ of its length from the top, and is to the weight of the water contained in the cylinder as twice the height to the perimeter.

22. The greatest height is when the c.g. of the whole solid coincides with the centre of the hemisphere.

h being the height of the cylinder, r its radius

$$\pi r^2 h \cdot \frac{h}{2} = \frac{2}{3}\pi r^3 \cdot \frac{3r}{8}.$$

$$\therefore\ h = r/\sqrt{2}.$$

23. Let V be the whole volume of the body.

σ, ρ, the intrinsic weights of the body and fluid.

$$V\sigma = P_1\rho_1 + (V - P_1)\,\rho = P_2\rho_2 + (V - P_2)\,\rho = P_3\rho_3 + (V - P_3)\,\rho.$$

$$\therefore\ \frac{\rho_2 - \rho_3}{P_1} + \frac{\rho_3 - \rho_1}{P_2} + \frac{\rho_1 - \rho_2}{P_3} = 0.$$

24. Volume of frustum $= \frac{7}{8}$ that of complete cone.

Volume of part immersed $= \frac{27}{64} - \frac{1}{8} = \frac{19}{64}$ that of cone.

∴ Density of cone : that of fluid $= 19 : 56$.

25. Expressing the same condition as in (22).

$$\frac{1}{3}\pi r^2 \cdot h \cdot \frac{h}{4} = \frac{2}{3}\pi r^3 \cdot \frac{3r}{8}.$$

$$\therefore\ h = \sqrt{3} \cdot r.$$

26. σ, ρ, being intrinsic weights of rods and fluid.

a the length of each rod, α the sectional area.

Moment of fluid pressures about hinge

$$=a a \cdot \rho \cdot \frac{a}{2} \cdot \frac{1}{\sqrt{5}} + a a \rho \left(\frac{a}{\sqrt{5}} + \frac{a}{2} \cdot \frac{2}{\sqrt{5}}\right) + \frac{a a \rho}{2}\left(\frac{a}{4} \cdot \frac{1}{\sqrt{5}} + \frac{a\sqrt{5}}{2}\right)$$

$$=a^2 a \rho \frac{31}{8\sqrt{5}}.$$

Moment of weight of rods about hinge

$$=a^2 a \sigma \left(\frac{1}{2\sqrt{5}} + \frac{2}{\sqrt{5}}\right) + a a \sigma \cdot \frac{a\sqrt{5}}{2} = a^2 a \sigma \sqrt{5}.$$

$$\therefore \ \sigma : \rho = 31 : 40.$$

27. Let the surface of the fluid meet AC in D.

Since the pressures of the fluid and the weight of the triangle act vertically through the c. g.'s of BCD, BAC respectively.

These c. g.'s are in a vertical line. $\therefore$ AC is vertical. And $\therefore$ perpendicular to BD.

And density of fluid : density of triangle

$$=AC : CD = AC : CB \cos C = \sin B : \sin A \cos C.$$

28. Since all the pressures on the curved surface make the same angle with the vertical, the whole pressure and the resultant pressure are proportional.

Now the resultant pressure is the weight of the cone, $\therefore$ the whole pressure is constant.

29. Let σ, ρ, ρ', be the intrinsic weights of the solid and of the two fluids, h the height of the cylinder, x the length of it in the lower fluid.

$$(h-x)\rho + x\rho' = h\sigma,$$

$$x = h \cdot \frac{\sigma - \rho}{\rho' - \rho}.$$

If ρ be increased, x is diminished, i.e. the cylinder rises. If x becomes $x+\delta$, the upward pressure on the cylinder is proportional to

$$(h-x-\delta)\rho + (x+\delta)\rho' = h\sigma + \delta(\rho' - \rho),$$

i.e. such as to urge the cylinder back towards equilibrium. $\therefore$ the equilibrium is stable.

30. Let the surface of the fluid intersect the rods in D, E.

Let P be the stress at B in a vertical direction, σ, ρ, the intrinsic weights of the rods and fluid.

Then $$P+\rho . CE=\sigma . BC.$$

And taking moments about B for the rod BC,

$$\sigma . BC . \frac{BC}{2}=\rho . CE\left(BE+\frac{CE}{2}\right)$$

$$\therefore\ \rho : \sigma=BC : CE\left(1+\frac{BE}{BC}\right)=BC^2 : BC^2-BE^2,$$

$$P=\rho\,\frac{BE . CE}{BC}.$$

Taking moments about A for the rod AB,

$$P . BA+\sigma . BA . \frac{BA}{2}=\rho . AD . \frac{AD}{2}.$$

Whence $$AD^2=CE(3BE+BC),$$

and $$AD^2 : AB^2=3\frac{\sigma}{\rho}-2+2\sqrt{1-\frac{\sigma}{\rho}}.$$

This being less than unity

$$\sigma : \rho<5 : 9.$$

Instead of introducing the stress at B, we might have taken moments about A for the equilibrium of the system, and about B for the rod BC.

31. Let the side AC be divided in the ratio $x : 1-x$, by the surface of the fluid in the first position.

Then $$\sigma=(1-x)^2.$$

In the second position the area immersed is $\overline{1-x^2}$ times that of the triangle.

$\therefore$ The pressure on the hinge = weight of displaced fluid

$$= (1 - x^2 - \text{weight of triangle}) \text{ area of triangle} - W$$

$$= W\left(\frac{1-x^2}{\sigma} - 1\right)$$

$$= 2W\frac{1-\sqrt{\sigma}}{\sqrt{\sigma}}.$$

32. $(W-P)^2 + Q^2$ is the square of the resultant pressure on the circular base, which is constant so long as its centre remains unmoved.

33. Let h be the height of the cone, r the radius of the base.

The resultant vertical pressure on the horizontal section through the axis is whr^2.

The weight of the contained fluid

$$= \tfrac{1}{2}w \,.\, \tfrac{1}{3}h \,.\, \pi r^2.$$

$\therefore$ The resultant vertical pressure on the curved surface

$$= (\text{weight of fluid in cone})\left(\frac{3}{\pi} - \frac{1}{2}\right).$$

34. Let a be the inclination of the axis to the vertical, h its length, A the sectional area, W the weight of fluid displaced.

Difference of pressure on ends

$$= g\rho \,.\, h\cos a \,.\, A = W\cos a.$$

Resultant horizontal pressure on the curved surface

$$= W\sin a\cos a.$$

$\therefore$ Resultant vertical pressure on the curved surface

$$= W\sin^2 a.$$

$\therefore$ The resultant is inclined to the vertical at an angle

$$\frac{\pi}{2} - a,$$

as is evident, since every part of it is perpendicular to the axis.

35. Volume not immersed : volume of cone

$$=1 : 2\sqrt{2}=1 : (\sqrt{2})^3.$$

Let ABC be a principal section of the cone, A the vertex, B being in the surface, ABC being an equilateral triangle.

Since the c. g.'s of the whole cone and of the unimmersed portion are in a vertical line, AC is vertical.

The volume of the unimmersed portion

$$=\frac{\sqrt{3}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{3}}\times \text{that of the whole cone.}$$

Since the area of the section perpendicular to AC is

$$\frac{\sqrt{3}}{2\sqrt{2}} \text{ of the base.}$$

Hence the given condition is satisfied.

36. If θ be the semi-vertical angle, r the radius of the base, h the height.

$$\pi r^2\cdot\frac{h}{4}=\frac{\pi r^2}{\sin\theta}\cdot\frac{h}{12},$$

$$\therefore \theta=\sin^{-1}\tfrac{1}{3}.$$

37. Let A be the area of the disc, r the radius of the sphere, σ its density, h the depth of water in the vessel.

Then $$Ah=\tfrac{4}{3}\pi r^3(1-\sigma).$$

If the sphere be in contact with the disc and only just immersed, $h=2r$, and the equation becomes

$$A=\tfrac{2}{3}\cdot\pi r^2(1-\sigma).$$

38. If θ is the inclination of the deck to the horizon

$$\tan\theta=\frac{5}{20\times 21},$$

and the restorative moment of fluid pressure $=9000\,h\sin\theta$, h being the metacentric height.

The moment of the 20 tons is $20\times 21\cos\theta$,

$$\therefore h=\tfrac{56}{25}=2{\cdot}24 \text{ feet.}$$

CHAPTER V.

EXAMINATION.

2. $$\frac{C}{5}=\frac{F-32}{9}=\frac{R}{4}, \text{ and } F=40^0,$$

$$\therefore R=3^0\tfrac{5}{9}, \qquad C=4^0\tfrac{4}{9}.$$

4. The required height $=\dfrac{13{\cdot}568}{5{\cdot}6}\times 30=72{\cdot}7$ inches nearly.

5. The pressure on each square inch is 1728 times as great as before. $\therefore$ the pressures on a side are as 1 : 12.

6. $$p=k\rho\,(1+at).$$

The volume is reduced to one-eighth, and $\therefore$ the density is increased 8-fold. $\therefore$ the pressure is $8\,(1+at)$ times as great. The surface of the smaller sphere is $\frac{1}{4}$ that of the larger,

$\therefore$ the pressure on it is $2\,(1+at)$ times that on the larger.

7. A small aperture made at the highest point of a siphon would cause the liquid to flow out of each arm.

9. The column would be longer, as the vertical distance between the top and bottom of it must remain the same.

10. $$\frac{C}{5}=\frac{F-32}{9}=\therefore\frac{F+C-32}{14}=-\frac{32}{14}\because F+C=0.$$

$$\therefore F=11^0\tfrac{3}{7}, \qquad C=-11^0\tfrac{3}{7}.$$

12. (i) No effect, if the top of the siphon is less than 30 inches above the openings of the ends.

(ii) The mercury flows out of that end which is lowest.

13. The pressure is increased or decreased by an amount obtained by multiplying the area of the surfaces by the increase or decrease of the atmospheric pressure.

14. The wood, since it displaces more air and therefore experiences a greater upward pressure.

15. (i) There will be no change except that part of the liquid may flow out of the new hole, if that hole be below the level of the liquid into which the shorter arm dips. If above that level, the liquid above the hole will flow back into the vessel.

(ii) The liquid above the hole flows on through the longer arm, that below it falls back into the vessel.

16. (i) To prevent the pressure of the steam forcing the tea out at the spout.

(ii) To admit air above the surface of liquid in a vessel, so that it may more readily flow out.

17. Required force $= \pi \,.\, 81 \,.\, \frac{15}{2} = 1909$ lbs. nearly.

18. At the same level, since the weight of the glass is equal to that of the mercury displaced.

19. Not unless the top of the siphon is so near the greatest height over which it can carry the liquid for which it is being used, that the fall in the barometer reduces the possible height below the actual.

20. The weight of the displaced air being diminished, the tension of the string will be increased.

21. The pressure is reduced in the ratio 3 : 2 by each stroke.

Now $$\left(\frac{2}{3}\right)^5 > \frac{1}{8} > \left(\frac{2}{3}\right)^6.$$

$\therefore$ the bladder is fully distended between the fifth and sixth strokes.

22. $$p = \tfrac{30}{12} \times 13{\cdot}59 \times \tfrac{1000}{16} g \text{ units of force},$$
$$\rho = {\cdot}0013 \times \tfrac{1000}{16} \text{ units},$$
$$\therefore k = \frac{5 \times 13{\cdot}59 \times 32}{2 \times {\cdot}0013} = 836308 \text{ about.}$$

Examples.

1. The volume is increased and $\therefore$ the density diminished in the ratio $1 : n^3$.

$\therefore$ the pressure is altered in the ratio $n^3 : 1 + at$.

2. The pressure being constant, the density varies inversely with the absolute temperature.

3. The formula of Art. (89) becomes in this case,

$$z = \frac{k}{g} \log \frac{W + W'}{W'},$$

where W is the weight of the fluid, and W' the weight of the piston + the pressure of the external air; and if Π be the pressure when ρ is the density,

$$\Pi = k\rho.$$

$\therefore$ length of cylinder occupied by fluid.

$$= \frac{\Pi}{g\rho} \log \left(1 + \frac{W}{W'}\right) = \frac{\Pi}{g\rho} \cdot \frac{W}{W'}, \text{ approximately,}$$

W being practically small compared with W'.

4. The new pressure would be nP if the temperature were constant. It is $\therefore$

$$nP \frac{1 + at'}{1 + at}.$$

5. Let W be the weight, A the area of the piston, $2a$ the length of the cylinder, x the height above the base at which the piston is in equilibrium, ρ the density of the air when the piston is in the middle.

Then $$\left[\frac{a}{x}-\frac{a}{2a-x}\right]k\rho A=W.$$

If we write $$k\rho A=mW,$$
the equation becomes
$$x^2-2ax+2ma(a-x)=0.$$

6. The increase of pressure is $k\rho at$ per unit area.

7. If σ be the density of mercury, it is $\frac{\sigma h}{\rho}$ the height of a barometer of the given liquid.

8. The pressures are as $h : h'$, and therefore the densities of the air are in this ratio.

9. Let Π be the external air-pressure, W the weight of the piston, A its area, λ the modulus of elasticity, a the natural length of the string, t the increase of temperature, x the increase of length of the string.

Increase of pressure of air within
$$=\left[\frac{a}{a+x}(1+\alpha t)-1\right]\times\text{original pressure.}$$

Increase of tension of string $=\lambda\frac{x}{a}$.

And original pressure $=\Pi A+W$.

$$\therefore\quad \frac{\lambda x}{a}=\frac{a\alpha t-x}{a+x}(\Pi A+W).$$

10. Let x be the fraction of the air immersed when the air is admitted.

$$x+(1-x)\cdot 0013=\tfrac{3}{4}.$$
$$\therefore\quad x=\tfrac{7487}{9987}.$$

The length above the surface is changed in the ratio $\cdot 25 : 1-x$
$$=\cdot 9987 : 1.$$

11. (i) It rises so long as the density of the air remains less than that of water.

(ii) It falls.

12. Suppose the water to rise x feet.

$$15 : 15 - x = 33\tfrac{3}{4} + 15 - x : 33\tfrac{3}{4}.$$

$$\therefore \quad 4x^2 - 255x + 900 = 0.$$

The lesser root is $x = 3\tfrac{3}{4}$.

The greater root is 60, which cannot apply to the present problem.

13. The hexagonal base contains six equilateral triangles, each one-fourth that into which the hexagon is formed.

$\therefore$ The pressure is increased in the ratio 6 : 4, i.e. 3 : 2.

14. The pressure must be increased to 8 times its former value, i.e. the glass must be immersed to a depth $7h$, where h is the height of the water barometer.

15. The increased pressure causes the air in the envelope to decrease in volume.

16. Let a be the depth to which the original open surface of the mercury is lowered, ρ, σ the densities of water and mercury, k, K the sectional areas of the tube.

If x be the increase of height of the mercury, x is made up of a rise of $Kx/(K+k)$ in the tube, and a fall of $kx/(K+k)$ in the reservoir.

$$\therefore \quad \rho\left[a + \frac{kx}{K+k}\right] = \sigma x.$$

$$\therefore \quad x = a\frac{\rho\left(1+\dfrac{K}{k}\right)}{\sigma\left(1+\dfrac{K}{k}\right) - \rho}.$$

If W be the weight of barometer and tube, W' the weight of an equal volume of water,

Tension of string $= W - W' + \frac{g\rho x k}{K+k}$.

If a be increased, x is increased, $\therefore$ tension is increased.

17. Let x be the distance of the piston from its former position, ρ the density of the atmosphere.

Then $$\frac{a}{a-x}\rho - \frac{a}{a+x}\rho = \rho \cos a.$$

Whence $$x = a\left[(1+\sec^2 a)^{\frac{1}{2}} - \sec a\right].$$

18. Let a be the length of the cylinder, b the length of it originally immersed, x the height of the water, Π the atmospheric pressure, h the height of the water-barometer.

$$\frac{a-b}{a-x}\Pi + \frac{x}{h}\Pi = \Pi$$

$$\therefore \quad x^2 - (a+h)\,x + bh = 0$$

determines the height x.

19. Let a be the length of each barometer, x, y the lengths of the tubes occupied by the air under a pressure equal to that of a length l of mercury.

$$h + \frac{x}{a-h}\,l = k + \frac{y}{a-k}\,l,$$

$$h' + \frac{x}{a-h'}\,l = k' + \frac{y}{a-k'}\,l,$$

$$\therefore \quad x : y = \frac{h-k}{a-k'} - \frac{h'-k'}{a-k} : \frac{h-k}{a-h'} - \frac{h'-k'}{a-h},$$

supposing the temperature the same at the two observations.

20. Let x cubic inches be the required volume.

The pressure is

$$\frac{32 - 5 + \frac{x}{12}}{32} \times \text{atmospheric pressure.}$$

$$\therefore \quad \frac{20 \times 32}{273+87} = \frac{\left(32 - 5 + \frac{x}{12}\right)x}{273+15}.$$

Whence $x = 17{\cdot}96\ldots$ cubic inches.

CHAPTER VI.

EXAMINATION.

1. THE pressure is increased threefold approximately.

$\therefore$ the air occupies one-third of its former volume.

2. The air will flow out, for its pressure is equal to the pressure of the water within the bell, i.e. is greater than that of the water at the top of the bell.

3. To a height not exceeding that of the mercurial barometer.

5. Area of piston $A = 64$ times that of the plunger C.

Pressure on plunger $= 4 \times 2 = 8$ lbs.

$\therefore$ pressure on piston $= 64 \times 8 = 512$ lbs.

6. The density is diminished by each stroke in the ratio 5 : 4.

Now $\left(\frac{4}{5}\right)^3 = \frac{64}{125}$, i.e. slightly exceeds $\frac{1}{2}$

and $\left(\frac{4}{5}\right)^4 = \frac{256}{625}$, i.e. is considerably less than $\frac{1}{2}$.

$\therefore$ the density is diminished one-half early in the fourth stroke.

9. The manometer has to register a pressure exceeding that of the atmosphere by three times.

$\therefore$ its length must be not less than $\frac{3}{2} \times 30 = 45$ inches.

[$\Pi' - \Pi = 2g\sigma x$, see Art. 114.]

11. Weight of water discharged per minute

$$=\pi \cdot \frac{1}{4} \cdot \frac{5}{2} \cdot 8 \cdot \frac{1000}{16} = \frac{5000\pi}{16} \text{ lbs.}$$

12. Weight of water discharged per minute

$$=\frac{\pi}{4} \cdot \frac{3}{2} \cdot 8 \cdot \frac{1000}{16} = \frac{3000\pi}{16} \text{ lbs.},$$

since the effective range of the piston is only

$$33 - 31\tfrac{1}{2} = \tfrac{3}{2} \text{ feet.}$$

EXAMPLES.

1. The density has become $\left(\frac{4}{5}\right)^5 = \frac{1024}{3125}$ of its original amount.

$\therefore$ $\frac{2101}{3125} = {\cdot}67232$ of the air has been pumped out.

2. The increased pressure, due to the escape of carbonic acid gas, would drive out some of the water and $\therefore$ the displacement being increased without increase of weight, the tension of the rope would be diminished.

3. Weight of water displaced $= \dfrac{P - P'}{W} \times W = P - P'$,

$\therefore$ the bell is in equilibrium, and as in Appendix II. Ex. 8 this equilibrium is unstable.

4. Suppose x feet occupied by air

$$\frac{x}{5} = \frac{33}{88 + x}.$$

Whence $\qquad x = 1{\cdot}83$ nearly.

Let V be the volume of the bell, nV the volume of air at atmospheric pressure, which is required to fill the bell;

then $\qquad (V + nV)\,33 = V\,(55 + 33 + 5) = 93\,V,$

$$\therefore \; 11n = 20.$$

5. After three strokes, the height of the mercury

$$=\left(\frac{3}{4}\right)^3\times 30=12\tfrac{2}{3}\text{ inches nearly.}$$

6. Let h be the height of the barometer in inch x the rise after four strokes.

The pressure in the tube after three strokes = tl due to a column $\frac{3h}{2}$ of mercury. $\therefore$ pressure in recei is due to the column $\frac{3h}{2}+5$;

but $$\frac{\rho_3}{\rho}=1+3\frac{B}{A},\quad \therefore\ \frac{B}{A}=\frac{1}{6}+\frac{5}{3h}.$$

$$\therefore\ \frac{\rho_4}{\rho}=1+4\frac{B}{A}=\frac{5}{3}+\frac{20}{3h}.$$

$$\therefore\ \frac{\frac{5h}{3}+\frac{20}{3}-x}{h}=\frac{15}{15-x}\quad\text{or}\quad\frac{15}{15-x}+\frac{x}{h}=\frac{5}{3}+\frac{20}{3h}.$$

If $h=30$ inches, $x=6{\cdot}1$ inches nearly.

7. h being the height of the water-barometer,

$\frac{m}{m-1}h$ is the depth to which the bell is sunk.

At the end of one second the depth is

$$\frac{m}{m-1}h+n\text{ feet},$$

$\therefore$ the amount of air to be pumped in

$$=\frac{\frac{m}{m-1}h+n}{\frac{m}{m-1}h}V-V=\left(1-\frac{1}{m}\right)\frac{n}{h}V.$$

8. Let the length of the stroke be x feet.

$$\frac{4x}{10}=\frac{33}{23}.$$

$$\therefore\ x=3\tfrac{27}{46}=3\text{ feet 7 inches about.}$$

9. Let V be the volume of the receiver, v that of the barrel. V_1, V_2 the volumes of the solid immersed in the water and air. σ, ρ, ρ' the densities of the solid and of the air before and after the first stroke of the piston.

The water rises $h\left(1-\frac{\rho'}{\rho}\right)$ feet.

If A be the area of the base of the receiver, its volume becomes

$$V-hA\left(1-\frac{\rho'}{\rho}\right)-(V_2-v').$$

where v' is the volume of the solid now immersed in water, which was formerly in air.

We have $\sigma(V_1+V_2)=V_1+\rho V_2=V_1+v'+\rho'(V_2-v')$.

Whence $\frac{\rho'}{\rho}=\dfrac{V-V_2}{V-V_2\dfrac{1-\rho}{1-\rho'}-hA\left(1-\dfrac{\rho'}{\rho}\right)+v}$ determines ρ'.

10. Let x be the depth to which the bell is immersed, y the depth of water in it.

$$\sigma h+\rho(x-y)=\sigma h'.$$

And
$$\frac{a}{a-y}=\frac{h'}{h};$$

hence
$$x=\frac{\sigma}{\rho}(h'-h)+a\frac{h'-h}{h'}=\left(\frac{\sigma}{\rho}+\frac{a}{h'}\right)(h'-h).$$

11. Let a be the length originally occupied by air,

h the height of the barometer,

ρ the density of the atmosphere,

ρ_n the density in the receiver after n strokes of the piston,

x the difference of height required.

$a+\frac{x}{2}$ is the length now occupied by air,

$$\therefore\quad \frac{ah}{a+\frac{x}{2}}-x=\frac{\rho_n}{\rho}h$$

or $$x^2+2a(x-h)+\frac{\rho_n}{\rho}h(2a+x)=0.$$

12. The greatest tension of the piston rod is the weight of a column of water equal in length to the vertical distance between the spout and the surface of the water in the reservoir, and whose sectional area is the area of the barrel of the pump.

13. Let V, v be the volumes of the receiver and barrel, A the area, w the weight of the valve, Π the atmospheric pressure. The pressure in the barrel when the fraction x of the nth descent remains to be accomplished is $\frac{1}{x}\left(\frac{V}{V+v}\right)^{n-1}\Pi$.

If the valve begins to open at this point, this pressure $=w/A$;

$$\therefore \quad x=\left(\frac{V}{V+v}\right)^{n-1}\frac{\Pi A}{w}.$$

14. Let σ be the density in the receiver at any time.

If $$\sigma\frac{h+\beta}{\beta} \not> \rho\frac{a}{a+h}$$

the two parts of the barrel will never communicate;

$\therefore$ the limiting density is $\rho\cdot\frac{a\beta}{(a+h)(\beta+h)}$.

15. The density after n strokes of the piston being ρ_n, let the fraction x of the stroke be completed when the valve opens. Then

$$\rho_n=\left(\frac{A}{A+B}\right)^n\rho;$$

$$B(1-x)\rho=B\rho_n;$$

$$1-x=\frac{\rho_n}{\rho}=\left(\frac{A}{A+B}\right)^n.$$

The pressure below the piston

$$=\frac{A}{A+Bx}\frac{\rho_n}{\rho}\times\text{atmospheric pressure.}$$

$\therefore$ tension of piston rod : atmospheric pressure on piston

$$=1-\frac{A}{A+Bx}\cdot\left(\frac{A}{A+B}\right)^n : 1$$

$$=1-\left(\frac{A}{A+B}\right)^n : 1-\left(\frac{A}{A+B}\right)^n\cdot\frac{B}{A+B}.$$

16. We must have $\dfrac{\frac{\Pi v}{v'}}{t}=\dfrac{\frac{\Pi v}{v'}+y}{t+x}$;

since the pressure within the bell is $\dfrac{\Pi v}{v'}$ and v' is unchanged,

$$\therefore\quad \frac{x}{t}=\frac{yv'}{\Pi v}.$$

17. The pressure $=\dfrac{AB}{BC}\times$ atmospheric pressure.

Increase of pressure $=\dfrac{AC}{CB}\times$ atmospheric pressure.

$\therefore$ if the pressure increases uniformly,

$AC : CB$ increases uniformly.

18. The pressure is doubled by 20 strokes of the condenser. It is then decreased in the ratio $\left(\frac{20}{21}\right)^{14}$ by 14 strokes of the pump.

Now $\qquad \left(\dfrac{20}{21}\right)^{14}=\cdot 50506\ldots=\text{about } \tfrac{1}{2}$,

i.e. the density is approximately restored to its original value.

CHAPTER VII.

Examination.

1. The weight of water displaced by the solid must be 7 lbs.

$\therefore$ its sp. gr. is to that of water as 5 : 7.

2. The weight of A displaced by the solid is 9 lbs.

The weight of B displaced is 7 lbs.

$\therefore$ the sp. gr.'s of A and B are as 9 : 7.

3. Equal weights of A and B occupy volumes

$$5-\frac{\pi}{256},\quad 5-\frac{2\pi}{256}$$

respectively.

$\therefore$ their sp. gr.'s are as $1280-2\pi : 1280-\pi$.

6. V being the volume of cork required, and $\frac{96}{1000\times 7\cdot 6}$ cubic feet being the volume of the iron,

$$\cdot 24\, V+\frac{96}{1000}=V+\frac{96}{1000\times 7\cdot 6};$$

$$\therefore\quad V=\tfrac{128}{1805}\text{ cub. ft.}$$

7. Equal volumes of the body, water, and spirit weigh 250, 210, and 200 grains respectively.

$\therefore$ the sp. gr. of the body is $\frac{25}{21}$;

that of spirit is $\frac{20}{21}$.

8. Let V be the volume of the hydrometer, W its weight, a the sectional area of the stem, s the sp. gr. of water.

W the weight of the hydrometer = that of a volume $(V - a\alpha)$ of water $= s(V - a\alpha)$.

$W + w$ = weight of a volume $(V + v' - a\alpha)$ of liquid A.

$\therefore$ sp. gr. of A : sp. gr. of water =

$$\frac{W+w}{V+v'-a\alpha} : s = W+w : W+sv'.$$

9. If s is sp. gr. of metal, and σ of spirit, the weight of the metal

$$= Vs = 200 + V(s-1) = 160 + V(s-\sigma),$$

from which $V = 200$, and $V\sigma = 160$,

so that $5\sigma = 4$.

10. Since weight of metal in water is 15 oz., and weight of metal and wood in water is 10 oz.;

$\therefore$ weight of water displaced by wood is 25 oz.;

$\therefore$ sp. gr. of wood $= \frac{20}{25} = \frac{4}{5}$.

Examples.

1. Whole weight of water displaced = 61 lbs.

The silver displaces $\frac{42}{10 \cdot 5} = 4$ lbs.

$\therefore$ the wood displaces 57 lbs.

$\therefore$ its sp. gr. = that of water.

2. If w be the weight of the material, $4w$ is the weight of the sinker in water.

$\therefore$ $2w$ is the weight of water displaced by the material.

$\therefore$ its sp. gr. is ·5.

3. The volume of the metal $=1-\left(\frac{8}{9}\right)^3$ cubic inches.

Its weight $=$ that of $1+\frac{4\cdot34}{2}=3\cdot17$ cubic inches of water.

$\therefore$ its sp. gr. $=\frac{729}{217}\times3\cdot17=10\cdot65$ nearly.

4. The weight of the wax $=1\cdot92$ grains.

That of an equal volume of water $=1\cdot8432$ grains.

$\therefore$ the weight of a quantity of water equal in volume to the crystal $=3\cdot3568$ grains.

$\therefore$ the sp. gr. of the crystal $=1\cdot876$ nearly.

5. The sp. gr.'s are as $7:7\frac{1}{2}=14:15$.

6. A quantity of liquid equal in volume to the solid weighs $1\frac{1}{2}$ ozs.

$\therefore$ sp. gr. of solid : that of liquid $=4:3$.

7. Let w_1, w_2, w_3, be the weights of the gold, the diamond, and a ruby respectively,

$$w_1+w_2+2w_3=44\tfrac{1}{4} \text{ grains},$$

$$\frac{w_1}{16\frac{1}{2}}+\frac{w_2}{3\frac{1}{2}}+\frac{2w_3}{3}=5\tfrac{1}{2}.$$

The weight of a ruby in water is 2 grains, $\therefore$ in air it is 3 grains. Hence we obtain

$$w_2=5\tfrac{1}{4} \text{ grains}.$$

8. Let x gallons of water be mixed with one of whiskey. Then

$$\cdot75+x=\cdot8\,(1+x);$$

$$\therefore\quad x=\cdot25.$$

$\therefore$ the price should be $\frac{4}{5}\,.\,16=12\frac{4}{5}$ shillings per gallon.

9. If W is the real weight of the body, and w, w' the real weights of the weighing pieces, we have

$$\frac{W}{\rho}(\rho-1)=\frac{w}{\rho'}(\rho'-1),$$

and
$$\frac{W}{\rho}\left(\rho-\frac{29}{30}\right)=\frac{w'}{\rho'}\left(\rho'-\frac{29}{30}\right).$$

The apparent weights of the body are w and w',

and since
$$\frac{1-\dfrac{1}{\rho}}{1-\dfrac{29}{30\rho}}=\frac{w}{w'}\cdot\frac{1-\dfrac{1}{\rho'}}{1-\dfrac{29}{30\rho'}},$$

it follows that

$$\frac{w'-w}{w}=\frac{\rho'-\rho}{(\rho-1)(30\rho'-29)}.$$

This is an increase or a decrease according as $\rho' >$ or $< \rho$.

10. Let w be the weight of the bottle, V its internal volume, v the volume of its material.

σ_a, σ_b, σ_c the sp. gr.'s of A, B, C.

$$w+V\sigma_a-(V+v)\,\sigma_b=A_b,$$
$$w+V\sigma_a-(V+v)\,\sigma_c=A_c,$$
$$w+V\sigma_b-(V+v)\,\sigma_c=B_c,$$
$$w+V\sigma_b-(V+v)\,\sigma_a=B_a,$$
$$w+V\sigma_c-(V+v)\,\sigma_a=C_a,$$
$$w+V\sigma_c-(V+v)\,\sigma_b=C_b.$$

$$\therefore\ A_b+B_c+C_a=A_c+B_a+C_b.$$

CHAPTER VIII.

EXAMINATION.

1. Pressure of mixture $= \dfrac{1728 \times 15 + 1 \times 60}{1729} = 15{\cdot}026$ lbs. per sq. inch.

3. Dew would be deposited on the furniture, &c., in the room.

9. Temperature of mixture $= \dfrac{3 \times 45 + 6 \times 90}{9} = 75°$.

10. The dryness of the air, and its capacity to absorb moisture owing to its low pressure, causes evaporation from the surface of the skin to go on more rapidly than moisture can be supplied from the subjacent tissues.

EXAMPLES.

1. When the volume is $V + V'$ and the temperature t, the pressure is

$$\frac{pV + p'V'}{V + V'}.$$

When the volume is changed to U and the temperature to t', the pressure becomes

$$\frac{273 + t'}{U} \cdot \frac{pV + p'V'}{273 + t}$$

$$= \frac{pV + p'V'}{U}(1 + a\,\overline{t' - t}) \text{ where } a = \tfrac{1}{273}.$$

2. The new pressures being ϖ, ϖ', and τ being the increase of temperature, (taking t as absolute temperature)

$$\frac{p}{t}=\frac{\varpi}{t+\tau}, \qquad \therefore\ \varpi-p=\frac{p\tau}{t};$$

$$\frac{p'}{t}=\frac{\varpi'}{t+\tau}, \qquad \therefore\ \varpi'-p'=\frac{p'\tau}{t}.$$

$\therefore$ the greater increase of pressure takes place where the original pressure was the greater.

Pressure at temperature zero $=\dfrac{p+p'}{2}\cdot\dfrac{273}{t}$;

or if t represent temperature on the centigrade scale,

$$\frac{p+p'}{2}(1-at).$$

3. If θ° be the final temperature,

$$2(100-\theta)\times\cdot12=(5+16\times\cdot2)(\theta-20)$$

$$\therefore\ \theta=22^\circ\tfrac{58}{211}.$$

4. In this case, with same notation,

$$\cdot12(120-\theta)+2\times\cdot1(90-\theta)=(6+10\times\cdot2)(\theta-10);$$

$$\therefore\ \theta=13^\circ\tfrac{53}{104}.$$

5. The space being saturated with vapour, one-half of it is condensed when the compression takes place.

Hence while the pressure of the air is doubled, that of the vapour remains unaltered.

Hence the difference of the observed pressures is the required original pressure of the air free from vapour.

6. The air in the ball-room was saturated with moisture in the form of vapour. This being suddenly frozen, became snow, when the temperature of the room was suddenly lowered below freezing-point.

7. The pressures (reduced to zero) in the air-space, are as

$$E_1(1-et_1)\ :\ E_2(1-et_2);$$

$\therefore$ a being the required ratio,

$$\frac{a^{t_1}}{a^{t_2}} = \frac{E_1(1-et_1)}{E_2(1-et_2)};$$

or
$$a = \left[\frac{E_1(1-et_1)}{E_2(1-et_2)}\right]^{\frac{1}{t_1-t_2}}.$$

8. The pressure on the under side of the piston never rises beyond twice its original value.

That above the piston is $\frac{3}{4}$ its original value.

$\therefore$ the weight of the piston $= 2 - \frac{3}{4} = \frac{5}{4}$ original pressure on it.

9. The vapour in the space above the water being condensed, the temperature of the water is above its boiling-point at the reduced pressure.

10. The freezing-point is lowered $\cdot 3^\circ$ $\therefore$ $\cdot 15$ of the heat required to raise the whole mass (if melted) through 1° is set free and expended in melting a portion of the mass.

$\therefore$ the mass melted is $\frac{\cdot 15}{79}$ = rather less than $\frac{1}{500}$th of the whole.

Or thus, the lowering of temperature $= (\cdot 0075)\,\cdot 40 = \cdot 3$; so that $-\cdot 3^0$ is the final temperature.

Let the portion x of the mass of ice M be melted; then

$$\begin{aligned} 79x &= \text{heat taken out of } x \text{ of water} \\ &\quad + \text{heat taken out of } M-x \text{ of ice} \\ &= x(\cdot 3) + \tfrac{1}{2}(M-x)(\cdot 3), \end{aligned}$$

taking the specific heat of water to be unity.

From this equation,

$$\frac{x}{M} = \frac{\cdot 3}{158 - \cdot 3} = \frac{3}{1577} = \frac{1}{526} \text{ nearly.}$$

CHAPTER IX.

EXAMPLES.

1. WE have (see Art. 159)

$$p=\rho\left[\tfrac{1}{2}\omega^2 QN^2 - g\,.\,ON\right];$$

$\therefore$ when ON remains constant, the difference of pressure varies as the difference of the values of QN^2.

2. Let l be the latus rectum of the vessel,

$\dfrac{2g}{\omega^2}$ is that of the liquid surface.

The volume of a paraboloid being one-half that of the cylinder on the same base and of the same height, the surface of the fluid must bisect the axis of the vessel.

$\therefore$ r being the radius of the rim,

$$\frac{r^2}{l}=2\,.\,\frac{r^2}{2g/\omega^2};$$

$$\therefore\quad \omega^2=g/l.$$

3. Let ω be the angular velocity of rotation,

h the height, r the radius of the cylinder,

x the depth immersed,

σ, ρ the densities of solid and fluid.

$$\sigma h=\rho\left[x-\tfrac{1}{2}\frac{\omega^2 r^2}{2g}\right];$$

$$\therefore\quad x=\frac{\sigma}{\rho}h+\frac{\omega^2 r^2}{4g}.$$

4. The common surface must be a surface of equal pressure in both liquids, which is possible, since the latera recta of the surfaces of equilibrium is independent of the densities of the fluids.

The surface is $\therefore$ a paraboloid of revolution.

5. If the paraboloidal surface touches the surface of the cone at the rim,

$$h^2 \tan^2 a = \frac{2g}{\omega^2} \cdot \frac{h}{2} \text{ and } \therefore \omega^2 = \frac{g}{h} \cot^2 a.$$

If $\omega < \cot a \sqrt{\frac{g}{h}}$, the depth of the vertex of the paraboloid below the rim of the cone

$$= r^2 \div \frac{2g}{\omega^2} = \frac{\omega^2 r^2}{2g}, \text{ where } r = h \tan a,$$

and the volume which runs over $= \frac{1}{2}\pi r^2 \cdot \frac{\omega^2 r^2}{2g}$.

If $\omega^2 = \frac{g}{h} \cot^2 a$, this is $\frac{1}{4}\pi h^3 \tan^2 a$.

If $\omega^2 > \frac{g}{h} \cot^2 a$, the liquid left in the cone will touch its surface in a circle of radius $2x \tan a$, such that

$$(2x \tan a)^2 = \frac{2g}{\omega^2} x.$$

Hence $x = g \cot^2 a / 2\omega^2$, and the volume of liquid left in the cone

$$= \frac{1}{3}\pi x^2 \tan^2 a \cdot 2x - \frac{1}{2}\pi x^3 \tan^2 a$$

$$= \frac{\pi}{48} \frac{g^3 \cot^4 a}{\omega^6}.$$

6. If x is the part of the axis not immersed, and a the radius of the bowl, the quantity which runs over is $\frac{1}{2}\pi a^2 x$, where $x = \frac{a^2\omega^2}{2g}$, and is therefore $\frac{\pi\omega^2 a^4}{4g}$.

7. Let P, Q be the free surfaces of the liquid in the tube, ANM the axis of rotation, C the centre of the ellipse. Then PQ passes through C, the tube being half full.

Draw PN, QM, CR, perpendicular to the axis of rotation

$$PN^2=\frac{2g}{\omega^2}.AN, \qquad QM^2=\frac{2g}{\omega^2}AM;$$

$$\therefore \quad \tan\theta=\frac{QM-PN}{AM-AN}=\frac{g}{\omega^2.CR}=\frac{g}{\omega^2 p}.$$

8. If we imagine the cylinder extended and the free surface completed to the curved surface of the cylinder, we see that the pressure on the upper end is equal to the weight of a volume

$$\tfrac{1}{2}\pi a^2.\frac{\omega^2 a^2}{2g} \text{ of liquid,}$$

i.e. it is $$g\rho\pi a^2.\frac{a^2\omega^2}{4g}.$$

That on the lower end $=g\rho\pi a^2\left[\dfrac{a^2\omega^2}{4g}+h\right]$.

The pressure on the curved surface

$$=\tfrac{1}{2}g\rho\left[h+\frac{\omega^2a^2}{2g}\right].2\pi a\left(h+\frac{\omega^2a^2}{2g}\right)-\tfrac{1}{2}g\rho.\frac{\omega^2a^2}{2g}.2\pi a.\frac{\omega^2a^2}{2g}$$

$$=g\rho\pi a h\left(h+\frac{\omega^2a^2}{g}\right).$$

9. If W be the weight of the bowl, a its radius,

W together with the weight of fluid in the bowl must be equal to the weight of fluid in a cylinder on the same base as the hemisphere and having a length a of its axis immersed in rotating fluid.

$$\therefore \quad W+\tfrac{2}{3}g\rho\pi a^2=g\rho\pi a^2\left(a+\frac{a^2\omega^2}{4g}\right);$$

$$\therefore \quad W=g\rho\pi a^2\left(\frac{a}{3}+\frac{a^2\omega^2}{4g}\right).$$

10. Let r be the distance of the cork from the axis, w, W the weights of the cork and of the water it displaces.

The fluid pressure is equivalent to a force vertically upwards, equal to W, and a force to the axis equal to $\frac{W}{g}\omega^2 r$.

Let T be tension of string, and θ its inclination to the vertical.

Then $T\cos\theta = W - w$, and $\frac{W}{g}\omega^2 r - T\sin\theta = \frac{w}{g}\omega^2 r$,

so that
$$\frac{W}{g}\omega^2 r\cos\theta = \frac{w}{g}\omega^2 r\cos\theta + (W-w)\sin\theta.$$

If a be the radius of the cylinder, and l the length of the string,
$$a = r + l\sin\theta.$$

These equations determine r and θ.

11. ω being the required velocity,
$$\tfrac{1}{2}h \,.\, \pi a^2 = \tfrac{1}{2}\frac{\omega^2 a^2}{2g} \,.\, \pi a^2 ;$$
$$\therefore \quad \omega = \sqrt{2gh}/a.$$

12. In this case we have
$$\tfrac{1}{2}\frac{\omega^2 h^2}{6g} \,.\, \pi \,.\, \frac{h^2}{3} = \tfrac{1}{2} \,.\, \frac{h}{3} \,.\, \pi \,.\, \frac{h^2}{3} ;$$
$$\therefore \quad \omega = \sqrt{\frac{2g}{h}}.$$

13. If $\frac{a^2}{4} \not> \frac{2g}{\omega^2}a$, i.e. $\omega \not< 2\sqrt{2g/a}$, the free surface does not intersect the side tubes and $\therefore$ no liquid flows out.

When $\omega = 2\sqrt{2g/a}$ the vertex of the paraboloidal free surface is at the point of intersection of the axis of rotation and the middle tube.

The whole pressure on tube at rest
$$= \tfrac{2}{3} g\rho a \times \text{surface of tube}.$$

The pressure on the middle tube when rotating is $\frac{1}{3}$ of what it was before since the area of the segment of the parabola $= \frac{2}{3}$ area of circumscribing square.

$\therefore$ whole pressure on tube $= \frac{4}{9} g\rho a \times$ surface of tube.

14. Let AB be a section of the surface of equal pressure which cuts the sphere (centre O) at right angles at P. OAN being the axis of rotation

$$PN^2 = \frac{2g}{\omega^2} AN = 3c \,.\, AN;$$

$$\therefore \quad c^2 \sin^2\theta = 3c \,.\, \frac{c\cos\theta}{2}.$$

The real root of which is given by $\cos\theta = \frac{1}{2}$.

Pressure at P = pressure at $A = \omega \,.\, AE$

$$= \omega\left(c - \frac{c\cos\theta}{2}\right) = \tfrac{3}{4}\omega c.$$

15. Suppose the free surface continued above the lid; volume of fluid above lid

$$= \tfrac{1}{2}\pi a^2 \,.\, \frac{\omega^2 a^2}{2g} - \tfrac{1}{2}\pi b^2 \,.\, \frac{\omega^2 b^2}{2g} - \pi(a^2 - b^2)\frac{\omega^2 b^2}{2g} = \frac{\pi\omega^2}{4g}(a^2 - b^2)^2.$$

The weight of this is the upward pressure on the lid, and the centre of pressure is the centre of the lid, i.e. its C.G. The whole volume of fluid in the cylinder

$$= \pi a^2 h - \tfrac{1}{2}\pi b^2 \frac{\omega^2 b^2}{2g} = \frac{\pi\omega^2}{4g}(a^4 - b^4);$$

$\therefore$ weight of lid : weight of fluid $= a^2 - b^2 : a^2 + b^2$.

16. Replace the spheres by vertical columns of liquid of length b.

The free paraboloidal surface will pass through the tops of these columns, and intersect the axis at the depth x below the centre, such that $a^2\omega^2 = 2g(b + x)$.

The pressure in the tube at the depth y below the centre

$$= \rho\left\{\tfrac{1}{2}\omega^2(a^2 - y^2) + g(y - x)\right\},$$

$$= \frac{\rho}{2}\left\{2b - \left(\omega y - \frac{g}{\omega}\right)^2\right\},$$

which is greatest when $y = g\omega^{-2}$.

CHAPTER X.

EXAMPLES.

1. THE greatest tension in each cylinder is at the base. We have $r^2h=r'^2h'$, $t=whr$, and $t'=wh'r'$;

$$\therefore \quad t : t'=r' : r.$$

2. $t=pr$, and $2t=p'.3r$,

$$\therefore \quad p : p'=3 : 2.$$

3. A bar one square inch in section can support 4000 lbs.; ∴ the greatest value of t is 400 lbs., and the greatest value of p is 80 lbs.

4. If r, r' be the radii, e, e' the thicknesses,

$$r^2e=r'^2e';$$

and the strengths are as

$$\frac{2e\tau}{r} : \frac{2e'\tau}{r'} \text{ or as } r'^3 : r^3.$$

5. Let $\frac{4}{3}\pi x^3$ be the volume of air, at atmospheric pressure Π, which is forced in.

Pressure with radius $b=\dfrac{x^3+a^3}{b^3}\Pi$;

and with radius $c \qquad =\dfrac{x^3+a^3}{c^3}\Pi$.

Let t be the tension with radius b, and t' with radius c; then

$$\Pi\left(\frac{x^3+a^3}{b^3}-1\right)=\frac{2t}{b}, \qquad \Pi\frac{x^3+a^3}{c^3}=\frac{2t'}{c},$$

and $$t : t' = b^2 - a^2 : c^2 - a^2,$$

so that $$\frac{x^3 + a^3 - b^3}{b^2} : \frac{x^3 + a^3}{c^3} :: b^2 - a^2 : c^2 - a^2,$$

which determines x.

6. The tension and pressure all along the band being the same, the curvature is the same, i.e. the band lies in a circular arc.

Let r be the radius of the circle at any time,

2α the angle subtended at the centre,

$2a$ the unstretched length of the band,

b the depth of the box, p the pressure inside;

$$a = r \sin \alpha; \quad t = (\Pi - p)\, r.$$

And $$2r\alpha = 2a\left(1 + \frac{t}{\lambda}\right) = 2a + 2ar\,\frac{\Pi - p}{\lambda};$$

$$\therefore \quad \alpha - \sin \alpha = \alpha \,.\, \frac{\Pi - p}{\lambda}.$$

When the band touches the bottom of the box,

$$r(1 - \cos \alpha) = b;$$

$$\therefore \quad \alpha = 2 \tan^{-1} \frac{b}{a};$$

and $$\Pi - p = \frac{\lambda}{a}\left[2 \tan^{-1} \frac{b}{a} - \frac{2ab}{a^2 + b^2}\right].$$

If $b > 2a$, the arc will gradually become a semicircle, and then the ends will be flattened against the vertical sides of the box, the free portion forming a semicircle.

7. At a depth h below the surface, the tension is $g\rho ha$.

Let $2b$ be the length of the side of the box,

r the radius of the curved portions of the membrane.

$$2\pi r + 4(2b - 2r) = 2\pi a;$$

$$\therefore \quad r = \frac{4b - \pi a}{4 - \pi};$$

$\therefore$ the tension at a depth $h = g\rho h \cdot \dfrac{4b - \pi a}{4 - \pi}$.

The change $= g\rho h \cdot \dfrac{4(a-b)}{4-\pi}$;

$\therefore$ change : original tension $= 4(a-b) : a(4-\pi)$.

8. If p, r be the increase of pressure and radius, a being the original radius, t the tension,

$$2\pi r = 2\pi a \frac{t}{\lambda}; \quad \therefore \quad t = \frac{\lambda r}{a}.$$

But $$t = p(a+r);$$

$$\therefore \quad r = \frac{pa^2}{\lambda - pa}.$$

If $r = a$, $\lambda = 2pa$.

9. The pressure at a depth $h = \rho\left(\frac{1}{2}\omega^2 a^2 + gh\right)$;

$$\therefore \text{ the tension} = \rho a\left(\tfrac{1}{2}\omega^2 a^2 + gh\right).$$

10. The pressure at the foot of the main

$$= 300 \times 62{\cdot}5 = 18750 \text{ lbs. per sq. foot.}$$

If τ be the required thickness in inches,

$$5 \times 2240\,\tau = \tfrac{18750}{1728} \times \tfrac{9}{2};$$

$$\therefore \quad \tau = \tfrac{1}{230} \text{ nearly.}$$

11. Let τ be the required thickness in inches

$$2000\,\tau = \frac{384 \times 62{\cdot}5}{1728} \times 6;$$

$$\therefore \quad \tau = \tfrac{1}{24} \text{ inch.}$$

MISCELLANEOUS PROBLEMS.

1. If D, E be the middle points of the sides, DE is parallel to AB, i.e. is horizontal.

$\therefore$ the line joining the centres of gravity of OAC and OBC is horizontal.

$$\begin{aligned} \therefore \text{ pressure on } OCA &: \text{pressure on } OCB \\ = \text{area of } OCA &: \text{area of } OCB \\ = \quad \sin 2B &: \sin 2A. \end{aligned}$$

2. When the centre of gravity is in the surface, the addition of a small quantity of water will obviously raise the centre of gravity of the whole.

Further, if a small quantity A be abstracted, the C.G. of the remainder B must be above the surface, for B, together with A, which certainly has its C.G. below the surface, form a mass whose C.G. is in the surface.

Hence, since either addition or subtraction of water raises the C.G., it must be in its lowest position when in the surface of the liquid.

3. Let h be the height of the cone, x the depth of the dividing plane below the vertex, $2a$ the angle of the cone.

Area of conical surface above the dividing plane

$$= \pi x^2 \tan^2 a \operatorname{cosec} a.$$

Its C.G. is at a depth $\frac{2}{3}x$.

$\therefore$ the whole pressure on it is $\frac{2}{3} g\rho\pi x^3 \tan^2 a \operatorname{cosec} a$.

The whole pressure on the whole cone is

$$\tfrac{2}{3}g\rho\pi h^3\tan^2 a\operatorname{cosec} a;$$

$$\therefore\ x^3=\tfrac{1}{2}h^3,\quad\text{or}\quad x=h/\sqrt[3]{2}.$$

If the vertex be downwards, the C.G. of the portion cut off is at a depth $\dfrac{h}{2}+\dfrac{x}{3}$;

$\therefore$ pressure on lower part $=\pi\rho g x^2\left(\dfrac{h}{2}+\dfrac{x}{3}\right)\tan^2 a\operatorname{cosec} a.$

Pressure on whole $=\tfrac{1}{3}\pi\rho g h^3\tan^2 a\operatorname{cosec} a$;

$$\therefore\ x^2\left(\frac{h}{2}+\frac{x}{3}\right)=\tfrac{1}{6}h^3,$$

$$\left(x-\frac{h}{2}\right)(x^2+2xh+h^2)=0,$$

$$x=\frac{h}{2}\text{ or }-h.$$

The solution $x=-h$ is obviously unsuited to the present problem.

4. The depth of the C.G. of the curved surface is

$$h\tan a\,.\cos a+\tfrac{1}{3}h\sin a=\frac{4h}{3}\sin a;$$

$\therefore$ whole pressure on it $=\tfrac{4}{3}\pi\rho g h^3\tan^2 a.$

Whole pressure on the base $=\pi\rho g h^2\tan^2 a\,.\,h\sin a$;

$\therefore$ pressure on curved surface : pressure on base

$$=4:3\sin a.$$

5. Let x_r be the depth of the point at which the rth line of division meets BC.

The triangle bounded by AB and this line has (if $AB=c$) an area

$$\tfrac{1}{2}cx_r,$$

and its C.G. is at a depth $\tfrac{1}{3}x_r$.

If p be the perpendicular from C on AB,

pressure on above triangle $=\frac{1}{6}cx_r^2$,

pressure on whole triangle $=\frac{1}{6}cp^2$;

$$\therefore \quad x_r^2=\frac{r}{n}p^2;$$

$\therefore$ the rth point of division of BC cuts off a fraction $\sqrt{r}/\sqrt{n}$ of the whole.

6. If ρ be the density of the solid, 2ρ, 3ρ, 4ρ are the densities of the fluids.

The mixture of equal volumes has a density 3ρ;

$\therefore$ $\frac{1}{3}$ of the solid is immersed in it.

The mixture of equal weights has a density

$$\frac{3}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}\rho=\frac{36}{13}\rho;$$

$\therefore$ $\frac{13}{36}$ of the solid is immersed in it.

7. The volume of the cone $=\frac{2}{3}\pi r^3=$ that of the hemisphere;

$\therefore$ the fluids are of equal density.

8. If the curved surface be turned upwards,

pressure on curved surface $=g\rho\,.\,2\pi r^2\,.\,\frac{r}{2}=g\rho\pi r^3$,

pressure on base $=g\rho\,.\,r\,.\,\pi r^2=g\rho\,.\,\pi r^3$;

$\therefore$ the pressures are equal.

If the curved surface be turned downwards, the pressure on it is

$$g\rho\,.\,\frac{3r}{2}\,.\,2\pi r^2=3g\rho\pi r^3$$

$=3$ times the pressure on the base.

9. Let x be the depth of the vertex, p the latus-rectum of the parabola.

Area immersed $=\frac{2}{3}x\,.\,\sqrt{px}$;

$$\therefore\ \sigma\,.\,\tfrac{2}{3}h\,.\,\sqrt{ph}=\rho\,.\,\tfrac{2}{3}x\,.\,\sqrt{px};$$

$$\therefore\ x=h\left(\frac{\sigma}{\rho}\right)^{\frac{2}{3}}.$$

10. The original volume of air when the sphere just closes the cylinder is

$$\pi r^2\,.\,h-\tfrac{2}{3}\pi r^3.$$

If ϖ be the pressure of the air in final equilibrium, Π the atmospheric pressure, x the distance of the centre of the sphere from the base of the cylinder,

$$(\varpi-\Pi)\,\pi r^2=W,$$

and volume of air finally $=\pi r^2\,.\,x-\tfrac{2}{3}\pi r^3$;

$$\therefore\ \frac{\pi r^2\,.\,x-\frac{2}{3}\pi r^3}{\pi r^2\,.\,h-\frac{2}{3}\pi r^3}=\frac{\Pi}{\varpi};$$

$$\therefore\ x=\tfrac{2}{3}r+\frac{\Pi}{\varpi}\left(h-\tfrac{2}{3}r\right),$$

$$\frac{\varpi}{\Pi}=1+\frac{W}{\pi r^2\,.\,\Pi};$$

$$\therefore\ x=\tfrac{2}{3}r+\frac{\pi r^2\,.\,\Pi\left(h-\frac{2}{3}r\right)}{W+\pi r^2\,.\,\Pi}$$

$$=\frac{\frac{2}{3}r\,W+h\pi r^2\,\Pi}{W+\pi r^2\,.\,\Pi}.$$

11. If l feet be the unit of length, the measure of the acceleration of gravity is $\frac{g}{l}\,.\,\frac{1}{4}$ and, as this is unity, it follows that $l=8$.

The mass of unit volume of the standard must be a pound, i.e. the mass of 512 cubic feet is one pound.

Now the mass of 512 cubic feet of water is 32000 lbs.;

$\therefore$ the ratio of the densities is 1 : 32000.

12. Let a be the semi-vertical angle of the cone, x the depth of axis immersed.

Volume immersed $=\frac{1}{3}x\,.\,\pi\,.\,x^2\tan^2 a$, and is constant.

Surface immersed $=\pi x^2 \tan^2 a \,.\, \operatorname{cosec} a$.

Hence this latter is least when $x \sin a$ is greatest.

Now $x^3 \tan^2 a$ is constant;

$\therefore$ $x \sin a$ is greatest when $\sin^3 a \cot^2 a$ is greatest,

i.e. $\sin a \cos^2 a$ is greatest.

$\sin a + \sin 3a$ is greatest when,

θ being a small change in a,

$$\sin(a+\theta) - \sin a = +\sin 3a - \sin 3(a+\theta),$$

$$\theta \cos a = -3\theta \cos 3a;$$

$$\therefore \ \cos a = -12\cos^3 a + 9\cos a;$$

$$\therefore \ \cos^2 a = \tfrac{2}{3} \ \text{ and } \ \tan^2 a = \tfrac{1}{2}.$$

13. If the plane be inclined at an angle θ to the horizontal, the depth of the C.G. of the lower half below the centre is $\frac{a}{2}\cos\theta$.

$\therefore$ the pressure on lower portion : pressure on upper portion

$$= a + \frac{a}{2}\cos\theta : a - \frac{a}{2}\cos\theta$$

$$= 2 + \cos\theta : 2 - \cos\theta.$$

If this ratio be $m : 1$,

$$2m - m\cos\theta = 2 + \cos\theta;$$

$$\therefore \ \cos\theta = 2\frac{m-1}{m+1}.$$

The greatest value of m is 3 when $\cos\theta = 1$, $\therefore \theta = 0$.

The least value of m is 1 when $\cos\theta = 0$, $\therefore \theta = \frac{1}{2}$.

14. Let $4a$ be the latus-rectum, x the distance of common surface from focus. The heights of the free surface above the focus are

$$r - 2a, \quad r' - 2a.$$

That of the common surface is $x-2a$;

$$\therefore\ \rho(r-x)=\rho'(r'-x);$$

$$\therefore\ x=\frac{\rho r-\rho' r'}{\rho-\rho'}.$$

15. The value of g in the new units is 32.

The weight of 1 unit volume of water is 4000 lbs.;

$\therefore$ the unit of weight is 125 lbs.

16. Let a be the thickness of each stratum.

The pressure at the bottom is

$$g\rho a[1+2+3+\ldots+n]=\frac{n(n+1)}{2}g\rho a.$$

At the bottom of the rth stratum, the density is $r\rho$ and pressure $\frac{r(r+1)}{2}g\rho a$.

The depth is $ra=h$;

$\therefore$ if $\frac{\rho}{a}=k$, the density will, when a is indefinitely diminished, vary as the depth

$$\text{and the pressure}=\frac{r(r+1)}{2}gk\,.\,\frac{h^2}{r^2}$$

$$=\tfrac{1}{2}gk\,.\,h^2,$$

where r is indefinitely increased.

17. Let $x\,.\,y$ be the heights of the surfaces of the two fluids above the base.

Then $\qquad x+y=\text{height of triangle},$

and $\qquad x : y=\rho' : \rho;$

$$\therefore\ x-y : x+y=\rho'-\rho : \rho'+\rho,$$

or, difference of heights of surfaces : height of triangle

$$=\rho'-\rho : \rho'+\rho.$$

18. The whole pressures are proportional to the pressures at the C.G.'s of the strata, i.e. on the rth piece the pressure is proportional to (see 16)

$$\left(\frac{r(r-1)}{2}+\frac{r(r+1)}{2}\right)g\rho a, \text{ i.e. to } r^2.$$

19. The whole pressure on the curved surface is (r being the radius)

$$\pi r(1^2+2^2+3^2+\dots n^2)g\rho a^2.$$

That on the base is

$$\pi r^2 . g\rho[1+2+3+\dots n]a;$$

$$\therefore \text{ whole pressure} = \pi g\rho ra\left[\frac{2n+1}{3}a+r\right]\frac{n(n+1)}{2}.$$

If the fluids be mixed the density is

$$\frac{n+1}{2}\rho;$$

$\therefore$ the pressure on the curved surface is

$$\pi r . g\rho . n^2a^2\frac{n+1}{2};$$

$\therefore$ it is increased in the ratio

$$\frac{n^2(n+1)}{2} : \frac{n(n+1)(2n+1)}{6},$$

or $$3n : 2n+1.$$

If the densities be $\sigma . \sigma+\rho \dots \tau+n\rho$,

the pressure at lowest point of nth stratum is

$$ga[\sigma+\sigma+\rho+\dots+\sigma+n\rho]$$
$$=gna\left[\sigma+\frac{n+1}{2}\rho\right].$$

Whole pressures on curved surfaces

$$=\pi r . g\sigma . n^2a^2+\pi r . g\rho a^2\frac{n(n+1)(2n+1)}{6}.$$

That on base is

$$\pi r^2 . g\sigma . na+\pi r^2 . g\rho a\frac{n(n+1)}{2}.$$

If
$$n\rho = kh,$$
$$na = h,$$
and n be increased while ρ and a are decreased indefinitely, the pressures become

$$\pi r \cdot g\sigma h^2 + \tfrac{1}{6}\pi r g k h^3 = \pi r g h^2 [\sigma + \tfrac{1}{6}kh],$$

and
$$\pi r^2 g\sigma h + \tfrac{1}{2}\pi r^2 g k h^2 = \pi r^2 g h [\sigma + \tfrac{1}{2}kh].$$

20. Let x, y be the depths of the vertex below the surface in the two positions, $2a$ the vertical angle of the cone.

Then volume immersed at first $= \frac{1}{3}x \cdot \pi x^2 \tan^2 a$;

$$\therefore\ y^3 = \frac{27}{8}x^3;$$

$$\therefore\ y = \tfrac{3}{2}x.$$

Volume within cone between these two positions

$$= \frac{19}{8} \cdot \tfrac{1}{3}\pi x^3 \tan^2 a.$$

Volume of a length $y - x \left(= \frac{x}{2}\right)$ of the cylinder is

$$\pi r^2 \cdot \frac{x}{2}.$$

Now $r^2 : x^2 \tan^2 a = 19 : 6$;

$\therefore$ volume of length $\frac{x}{2}$ of cylinder $= \frac{19}{12}\pi x^3 \tan a =$ twice volume of this portion of cone; $\therefore$ the surface of water in cylinder rises just sufficiently to keep equilibrium.

21. Let x be the distance of the C.G. of the surface from the axis, θ the angle made by the perpendicular from the C.G. on the axis with the vertical, S the area of the surface. The depression of the C.G. in turning through a right angle is

$$x(\cos\theta + \sin\theta).$$

In turning through another it is

$$x(\cos\theta - \sin\theta);$$

$$\therefore \quad g\rho S\, x(\cos\theta + \sin\theta) = A,$$

$$g\rho S\, x(\cos\theta - \sin\theta) = B.$$

The difference between the greatest and least pressures is $g\rho S \,.\, 2x \,. = \sqrt{2(A^2+B^2)}$.

22. Let $2h$ be the height, a the semi-vertical angle of the cone.

Area of surface of frustum $= \pi \,.\, 3h^2 \,.\, \tan^2 a \,.\, \operatorname{cosec} a$.

Its C.G. is at a depth x below the vertex where

$$3 \,.\, x + 1 \,.\, \tfrac{2}{3}h = 4 \,.\, \tfrac{4}{3}h;$$

$$\therefore \quad x = \tfrac{14}{9}h;$$

$\therefore$ whole pressure on curved surface

$$= g\rho \tfrac{14}{9}h \,.\, \pi \,.\, 3h^2 \tan^2 a \,.\, \operatorname{cosec} a$$

$$= \tfrac{14}{3}\pi g\rho h^3 \tan a \sec a.$$

Pressure on base $= g\rho \,.\, 2h \,.\, \pi \,.\, 4h^2 \tan^2 a$;

$$\therefore \quad \tfrac{14}{3} \operatorname{cosec} a : 8 = 7 : 6;$$

$$\therefore \quad \operatorname{cosec} a = 2;$$

$$\therefore \quad a = 30^\circ.$$

23. The weight of the contained fluid is

$$g\rho \,.\, \frac{7}{8} \,.\, \frac{2h}{3} \,.\, \pi \,.\, 4h^2 \tan^2 a$$

$$= \tfrac{7}{3}\pi g\rho h^3 \tan^2 a = W \text{ (Say)}.$$

The upward pressure on the curved surface

$$= \tfrac{14}{3}\pi g\rho h^3 \tan^2 a \operatorname{cosec} a \,.\, \sin a = \tfrac{14}{3}\pi g\rho h^3 \tan^2 a.$$

The upward pressure on the cover

$$= g\rho \,.\, \pi \,.\, h^2 \tan^2 a \,.\, h;$$

$\therefore$ whole upward pressure of fluid $= \tfrac{17}{3}\pi g\rho h^3 \tan^2 a = \tfrac{17}{7} W$;

$\therefore$ if the weight of the vessel be less than $\frac{17}{7}W$ it will be lifted.

24. If r be the radius of the cylinder, h the height of the cone, the volume immersed is

$$\frac{1}{8} \cdot \frac{h}{3} \cdot \pi r^2 = \tfrac{1}{24} \cdot \pi r^2 \cdot h.$$

If the surface of the fluid in the cylinder rise a distance x, the volume of the slice x of the cylinder outside the cone + that of a slice x of the cone $= \frac{1}{8}$ vol. of cone $=$ vol. of a slice $\frac{1}{24}h$ of the cylinder;

$$\therefore\ x = \tfrac{1}{24}h.$$

25. Let a be the depth of the fluid, h the height of the cone. Whole pressure on curved surface of cone

$$= \text{area of surface} \cdot (a - \tfrac{1}{3}h) \cdot g\rho\,;$$

$$\therefore\ \text{vertical pressure} = \text{area of base} \cdot g\rho\,(a - \tfrac{1}{3}h),$$

$$\text{weight of cone} = \text{area of base} \cdot g\sigma \cdot \tfrac{1}{3}h.$$

Upward pressure on $B =$ vol. of $B \cdot g(\rho - \sigma)$;

$$\therefore\ \text{we must have vol. of } B = \text{vol. of cone } \frac{\rho\left(\dfrac{3a}{h} - 1\right) + \sigma}{\rho - \sigma}.$$

26. The curve of buoyancy is a similar and similarly situated concentric ellipse, and in whatever position the ellipse is held, with its centre in the surface, the resultant fluid pressure is in the direction of the normal at the lowest point of the curve of buoyancy.

If the axis is horizontal, and the ellipse then slightly displaced, the normal at the lowest point of the buoyancy curve will intersect the minor axis above the centre of the ellipse. The equilibrium is therefore stable; and, similarly, the equilibrium is unstable when the axis is vertical.

Or we can reduce the problem to the case of an elliptic lamina, resting with its plane vertical on a horizontal plane, and then treat the question in the same manner. See Art. 66.

27. If T be tension of each string, a length of side, the moment of forces tending to keep a face in position

$$=T\sqrt{3}\times\text{height of tetrahedron}=T\sqrt{2}\,.\,a.$$

The centre of pressure on a face is halfway down.

The pressure is $g\rho\,.\,\frac{1}{3}$ height $\times\ \dfrac{a^2\sqrt{3}}{4}=$ weight of fluid.

$$\therefore\quad \frac{\sqrt{3}}{4}\text{ weight of fluid}=T\sqrt{2}\,.\,a.$$

$$\therefore\quad T:W=\sqrt{3}:4\sqrt{2}.$$

28. Let w_1, w_2,... be the weights and ρ_1, ρ_2,... the densities. When totally immersed they will be able to rest if the weight of the displaced liquid is equal to the sum of their weights; hence, if σ is the density of the liquid,

$$\sigma\,.\,\Sigma\,(w/\rho)=\Sigma\,(w).$$

If they are required to rest in any position, we must in addition have their C.G. coincident with the C.G. of the displaced liquid, i.e. with the C.G. of homogeneous bodies of the same volume.

Hence if x_1, x_2,... are their distances from a fixed point in the rod, we must have

$$\frac{\Sigma\,(wx)}{\Sigma\,(w)}=\frac{\Sigma\left(\dfrac{w}{\rho}x\right)}{\Sigma\left(\dfrac{w}{\rho}\right)}.$$

Applying this condition,

$$\frac{W_1x-W_3y}{W_1+W_2+W_3}=\frac{\dfrac{W_1}{\rho_1}x-\dfrac{W_3}{\rho_3}y}{\dfrac{W_1}{\rho_1}+\dfrac{W_2}{\rho_2}+\dfrac{W_3}{\rho_3}},$$

$$x\left[W_1W_2\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)+W_1W_3\left(\frac{1}{\rho_3}-\frac{1}{\rho_1}\right)\right]$$
$$=y\left[W_1W_3\left(\frac{1}{\rho_1}-\frac{1}{\rho_3}\right)+W_2W_3\left(\frac{1}{\rho_2}-\frac{1}{\rho_3}\right)\right],$$

$$\frac{x}{W_3}\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)+\frac{y}{W_1}\left(\frac{1}{\rho_3}-\frac{1}{\rho_2}\right)=\frac{x+y}{W_2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_3}\right).$$

29. It is clear that one position of equilibrium is when the rod is vertical.

If there is another let θ be the inclination of the rod to the vertical.

then if ρ, ρ' are the densities of the lower and upper liquids, and σ the density of the rod,

$$\tfrac{1}{2}\rho c^2 \sec^2\theta + \rho'(a - c\sec\theta)\frac{a + c\sec\theta}{2} = \sigma\frac{a^2}{2},$$

$$\therefore \quad \cos^2\theta = \frac{c^2}{a^2}\cdot\frac{\rho-\rho'}{\sigma-\rho'},$$

so that there is another position of equilibrium if

$$\sigma > \rho', \text{ and } c^2(\rho-\rho') < a^2(\sigma-\rho').$$

Considering the vertical position, give the rod a slight displacement ϕ;

Then the moment of the forces about the lower end varies as

$$\rho'(a^2 - c^2\sec^2\theta) + \rho c^2 \sec^2\theta - \sigma a^2,$$

or as

$$c^2(\rho-\rho')\sec^2\theta - a^2(\sigma-\rho'),$$

and this is positive if

$$c^2(\rho-\rho') > a^2(\sigma-\rho'),$$

which is the condition that the vertical position is the only one possible.

If the moment is negative, the inclined position is one of stable equilibrium.

30. Let a be the length of the cylinder, x the portion of it filled with air, Π the atmospheric pressure, h the height of the water-barometer.

The pressure of the air is $\dfrac{a}{2x}\Pi$.

$\dfrac{3a}{4} - x$ is difference of heights of water inside and outside.

$$\therefore \quad \frac{a}{2x}\Pi + \frac{\frac{3a}{4} - x}{h}\Pi = \Pi.$$

$$\therefore \quad 2ah + 3ax - 4x^2 = 4xh,$$

$$4x^2 + x(4h - 3a) - 2ah = 0.$$

The positive root of this equation gives the required value of x.

31. Let $3a$, $3b$ be the lengths of the sides of the parallelogram. The lines of division trisect the opposite sides of the figure.

The depths of the C.G.'s of the triangular portions are

$$\tfrac{2}{3} \cdot \left\{3\sqrt{a^2+b^2} - 2b \cdot \frac{b}{\sqrt{a^2+b^2}}\right\} = \tfrac{2}{3}\,\frac{3a^2+b^2}{\sqrt{a^2+b^2}},$$

and $$\tfrac{2}{3}\,\frac{3b^2+a^2}{\sqrt{a^2+b^2}}.$$

The areas are each $3ab$.

$$\therefore \quad P_1 = g\rho \,.\, 3ab \,.\, \tfrac{2}{3} \,.\, \frac{3a^2+b^2}{\sqrt{a^2+b^2}} = 2g\rho\, ab \,.\, \frac{3a^2+b^2}{\sqrt{a^2+b^2}},$$

$$P_3 = \qquad 2g\rho\, ab \,.\, \frac{3b^2+a^2}{\sqrt{a^2+b^2}},$$

$$P_1 + P_2 + P_3 = g\rho \,.\, 9ab \,.\, \tfrac{3}{2}\sqrt{a^2+b^2} = \tfrac{27}{2}\, g\rho\, ab\sqrt{a^2+b^2},$$

and $$P_1 + P_3 = 8g\rho\, ab \,.\, \sqrt{a^2+b^2},$$

$$\therefore \quad P_2 = \tfrac{11}{2}\, g\rho\, ab\sqrt{a^2+b^2},$$

$$\therefore \quad P_1 : P_2 : P_3 = 4(3a^2+b^2) : 11(a^2+b^2) : 4(a^2+3b^2),$$

and $$16P_2 = 11(P_1 + P_3).$$

32. Let ρ be the density of the liquid, h the height of the cone, r the radius of the base, so that $r = h\tan\alpha$.

The whole pressure on the curved surface is

$$g\rho \,.\, \tfrac{2}{3}h \,.\, r^2 \operatorname{cosec} \alpha.$$

The pressure on the base is $g\rho h\pi r^2$.

$$\therefore \quad P = g\rho\pi r^2 h\left[\frac{2+3\sin a}{3\sin a}\right],$$

$$P' = g\rho \,.\, \tfrac{1}{3}h \,.\, \pi r^2,$$

$$\therefore \quad \frac{P}{P'} = \frac{2+3\sin a}{\sin a}.$$

In the second case the depth of C.G. of surface is diminished in the ratio of 1 : cos θ.

$$\therefore \quad \frac{P}{P'} = \frac{2+3\sin a}{\sin a} \,.\, \cos\theta,$$

P' remaining unchanged.

33. The centre of pressure of any face divides the height in the ratio 3 : 1.

If a be the length of an edge,

Height of tetrahedron $= \sqrt{\tfrac{2}{3}} \,.\, a$.

Height of a face $= \dfrac{\sqrt{3}}{2} a$.

Let W, w be weights of the fluid and a face.

The pressure on a face

$= g\rho \times \tfrac{2}{3}$ height of tetrahedron $\times$ area of a face $= 2W$.

$\therefore \quad w \,.\, \dfrac{1}{9} \,.\, \dfrac{\sqrt{3}}{2}\, a$ must be not less than $2W \,.\, \dfrac{1}{4} \,.\, \dfrac{\sqrt{3}}{2}\, a$,

i.e. w not less than $\dfrac{9W}{2}$.

34. Let r be the radius of the cylinder, W its weight, ϖ the pressure of the vapour, Π that of the atmosphere.

If $(3\varpi - \Pi)\pi r^2 < W$, the vapour will be condensed

i.e. W must be not $> (3\varpi - \Pi)\pi r^2$.

35. The pressure will increase most in the larger bulb and therefore there will be a movement of the index.

36. Let x be the depth of air in the inverted cone when vertex is in the surface.

Its pressure is

$$\frac{a^3}{x^3}\Pi.$$

$$\therefore \quad \Pi + \frac{x}{h}\Pi = \frac{a^3}{x^3}\Pi,$$

or
$$1 + \frac{x}{h} = \frac{a^3}{x^3},$$

V being the volume of the cone.

$$W = g\rho \cdot \frac{x^3}{a^3}V. \quad = \frac{\Pi}{h}\frac{x^3}{a^3}V = \frac{\Pi}{x+h}V.$$

$$(m+1)W = g\rho V = \frac{\Pi}{h}V$$

$$= \frac{x+h}{h}W.$$

$$\therefore \quad m = \frac{x}{h},$$

$$\therefore \quad 1 + m = \frac{a^3}{m^3h^3} \text{ or } \frac{a}{h} = m\sqrt[3]{1+m}.$$

37. A volume $2nAl$ is pumped out per minute.

$\therefore$ $\dfrac{2nAl}{B}$ is the velocity per minute with which it issues.

38. Let a be the length of the cylinder, x the length of it filled with air at any time, h the height of the water barometer.

If
$$\frac{a}{x}\Pi > \Pi + \frac{a-x}{h}\Pi,$$

water will flow over,

$$\text{i.e. if } \frac{a}{x} + \frac{x}{h} - \frac{a}{h} > 1.$$

The limit is reached when the inequality becomes an equality,

$$\text{i.e. } x^2-(h+a)\,x+ah=0.$$

$$(x-h)(x-a)=0.$$

$$\text{i.e. } x=a \text{ or } x=h.$$

i.e. if once started, water may be poured in till its depth equals the height of the water barometer before any flows over.

If $x<h$ the piston will not begin to sink at all. The water runs over from the beginning.

39. The pressure in the interior cylinder is $\frac{a}{y}\Pi$,

$$\frac{a}{y}h+(a-y)=h+a-x. \quad \therefore \quad \frac{a-y}{y}h=y-x.$$

$$x-\frac{x-y}{2}=\frac{a}{2}.$$

$$\therefore \quad x+y=a,$$

$$\therefore \quad \frac{y}{x}(y-x)=h.$$

$$(a-x)(a-2x)=xh,$$

$$\therefore \quad 2x^2-x(3a+h)+a^2=0,$$

$$4x=3a+h\pm\sqrt{a^2+6ah+h^2},$$

$$\therefore \quad 4y=a-h\mp\sqrt{a^2+6ah+h^2}.$$

40. If $4a$, πa be the sides,

The area is $4\pi a^2$.

$\therefore$ the parts external to both semi-circles − the part common to both

$$=4\pi a^2-\pi\,.\,4a^2=0.$$

$\therefore$ the area of the common part = the area of the parts external to both.

And in the given position the C.G.'s of the one part and of the others taken together are each at the centre of the rectangle.

$\therefore$ the pressures are the same.

41. Let C be the centre of the rectangular hyperbola.

PN a perpendicular on the transverse axis.

PTt the tangent at P to the curve, cutting the axes in T, t.

The horizontal sections of the columns of water standing on an element of surface near P in the two cases are as

$$Ct : CT.$$

The heights of the columns are PN and CN,

$\therefore$ the pressures are as

$$PN \,.\, Ct : CN \,.\, CT = BC^2 : AC^2,$$

i.e. they are equal.

Thus since the pressures on every small portion of the area are the same in the two cases, the pressures are equal on any finite area.

42. Let a be the semi-vertical angle, $2h$ the height of either cone.

The area of the surface of the lower cone is

$$4\pi h^2 \tan a \,.\, \sec a.$$

$\therefore$ the whole pressure on it is $\frac{28}{3} g\rho\pi h^3 \sin a$.

$\therefore$ the resultant pressure is $\frac{28}{3} g\rho\pi h^3 \tan^2 a$.

The weight of the fluid in the upper cone is

$$\tfrac{1}{3} g\rho\pi h^3 \tan^2 a.$$

If W be the weight of fluid either cone can contain, w the weight of either cone,

$$W=\frac{8}{3}g\rho\pi h^3\tan^2 a.$$

$$2w+\tfrac{1}{3}g\rho\pi h^3\tan^2 a=\frac{28}{3}g\rho\pi h^3\tan^2 a,$$

$$\therefore\ 2w=\frac{27}{3}g\rho\pi h^3\tan^2 a=\frac{27}{8}W.$$

$$\therefore\ w : W=27 : 16.$$

43. Let a be the side of the square and let $CE=b$.

The whole pressure on the square $=\frac{1}{2}wa^3$.

The whole pressure on the triangle BCE is

$$\tfrac{1}{2}ab\,.\,\tfrac{2}{3}wa=\tfrac{1}{3}wa^2b\,;$$

$$\therefore\ \tfrac{1}{3}b=\tfrac{1}{4}a\,;\quad \therefore\ b=\tfrac{3}{4}a.$$

$\therefore$ the distance of the centre of pressure of BCE from BC

$$=\tfrac{3}{4}\,.\,\tfrac{3}{8}a=\tfrac{9}{32}a.$$

Its distance from $CD=\frac{1}{4}a$.

The centre of pressure of the square is $\frac{1}{3}a$ from CD;

$\therefore$ the distance between these two $=a\sqrt{(\frac{1}{12})^2+(\frac{7}{32})^2}$

$$=\frac{a}{96}\sqrt{505}.$$

Since the centre of pressure of the other part of the square and that of the triangle must be equidistant from the centre of pressure of the whole square, the distance between the two centres of pressure

$$=\frac{a}{48}\sqrt{505}.$$

44. Let a be the side of the square, θ the inclination at which the moveable side is inclined to the horizontal, and W its weight;

then, taking moments about the hinge,

$$W \cdot \frac{a}{2} \cos\theta = wa^2 \cdot \frac{a}{2} \sin\theta \cdot \frac{a}{3};$$

$$\therefore \quad \tan\theta = \frac{3W}{wa^3}.$$

45. Let $\theta+\alpha$, $\theta-\alpha$ be the inclinations of the two radii to the radius in the surface, a the radius.

The C.G. of the sector is on the bisecting radius at a distance $\frac{2}{3} \cdot \frac{a \sin\alpha}{\alpha}$ from the centre;

$\therefore$ its depth is $\frac{2}{3} \frac{a \sin\alpha \sin\theta}{\alpha}$;

$\therefore$ the pressure is $w \cdot \frac{2}{3} \frac{a \sin\alpha \sin\theta}{\alpha} \cdot \alpha a^2$

$$= \tfrac{2}{3} wa^3 \sin\alpha \sin\theta = \tfrac{1}{3} wa^3 \{\cos(\theta-\alpha) - \cos(\theta+\alpha)\}.$$

The pressure on the whole $= \frac{2}{3} wa^3$.

If $\theta = \alpha$ and the pressure on the sector is $\frac{1}{6} wa^3$,

$$\tfrac{1}{2} = 1 - \cos 2\theta = 2\sin^2\theta;$$

$$\therefore \quad \sin\theta = \tfrac{1}{2}; \quad \therefore \quad \theta = 30^\circ.$$

$\therefore$ the bounding radius makes an angle of 60° with the radius in the surface.

46. Since the cylinder may be divided into narrow vertical strips, the centre of pressure of each of which divides its length in the ratio 2 : 1, the centre of pressure of the whole is at this same depth.

It is also obviously at the C.G. of the section of the cylindrical surface at this depth.

47. In each case the centre of pressure must be at the same depth as that of the triangle intercepted by the two given planes on the vertical plane through the axis perpendicular to the given radius;

$\therefore$ it divides the height in the ratio 3 : 1 in the first case.

It bisects the height in the second case.

48. Let h be the height of the pyramid, A the area of the base, d the depth of the centre of the base.

The horizontal pressure on the inclined surfaces = that on the base $= wAd$.

The vertical pressure on the inclined surfaces = weight of fluid displaced $= \frac{1}{3}wAh = W$ (say);

$$\therefore \text{ Resultant pressure} = W\sqrt{1+\frac{9d^2}{h^2}}.$$

And it is inclined at an angle $\tan^{-1}\frac{3d}{h}$ to the vertical.

If the base be inclined to the vertical at an angle θ,

The horizontal component of the pressure

$$= g\rho Ad\cos\theta = \frac{3Wd}{h}\cos\theta.$$

The vertical component $= W \pm \frac{3Wd}{h}\sin\theta$.

$$\therefore \text{ the resultant} = W\sqrt{1 \pm \frac{6d}{h}\sin\theta + \frac{9d^2}{h^2}}.$$

And it is inclined to the vertical at an angle

$$\tan^{-1}\frac{3d\cos\theta}{h \pm 3d\sin\theta}.$$

The upper sign applies if the vertex is depressed below the centre of the base, the lower in the contrary case.

49. If h be the perpendicular distance of the base from the vertex, a the radius of the base, d the depth of the centre of the base,

Horizontal pressure on curved surface $= w\pi a^2 . d$.

Vertical pressure $=\frac{1}{3}w\pi a^2 . h = W$, the weight of fluid displaced;

$$\therefore \text{ Resultant pressure} = W\sqrt{1+\frac{9d^2}{h^2}}.$$

50. The resultant pressure on the fluid is equal to its weight and acts vertically through its centre of gravity, and this is equal and opposite to the resultant pressure on the curved surfaces of the cone.

Now the C.G. of the fluid is in a line joining the vertex to the centre of the base and divides that line in the ratio 1 : 3.

$\therefore$ if a vertical be drawn through the centre of the base to meet the curved surface, and the line joining its point of intersection with that surface to the vertex be divided in the ratio 1 : 3, the point thus obtained is the centre of pressure required, while the pressure is equal to the weight of the contained fluid.

51. Since the diameter through the point of contact bisects all horizontal chords, the centre of pressure always lies in that diameter.

Let θ be the angle between that diameter and the horizontal.

If QVQ' be any ordinate, PV being the diameter,

$$QV^2 = 4SP . PV = 4AS \operatorname{cosec}^2\theta . PV$$
$$= 4AS . \operatorname{cosec}^3\theta . (\text{depth of } V).$$

$\therefore$ if the area be divided by a number of horizontal lines which remain always at the same depths, the portions between any two such consecutive lines are increased in area in the ratio $\operatorname{cosec}^{\frac{3}{2}}\theta : 1$.

$\therefore$ all the pressures being increased in the same proportion, the centre of pressure remains at the same depth.

52. The resultant horizontal pressure on the part described is equal to that on the corresponding portion of the triangular plane face of the half cone.

Now the depth of the C.G. of a cone and of the centre of pressure of an isosceles triangle whose vertex is in the surface, is in each case $\frac{3}{4}$ of the height, and they are therefore in this case in the same horizontal line.

From this the required result is obvious, on drawing a figure.

53. The contained air is cooled by expansion and any moisture in it becomes condensed.

54. The pressure at all points being normal to the surface, the resultant pressure must act through the centre of the sphere.

The horizontal component = the pressure on the diametral plane $= w\pi a^3$,

and the vertical component $= \frac{2}{3}w\pi a^3 = W$ (say);

$\therefore$ the resultant $= \frac{1}{2}W\sqrt{13}$,

and the inclination to the vertical $= \tan^{-1}\frac{3}{2}$.

55. If 2θ is the angle between the planes, and $2a$ the angle of the cone,

the pressure on each triangular face $= w \,.\, \frac{1}{2}h^2 \tan a \,.\, \frac{1}{3}h$;

$\therefore$ resultant horizontal pressure $= \frac{1}{3}wh^3 \tan a \sin\theta$,

the vertical pressure $= \frac{1}{3}wh\theta h^2 \tan^2 a = W$;

$\therefore$ resultant pressure $= W \,.\, \sqrt{1 + \theta^2 \tan^2 a \operatorname{cosec}^2\theta}$,

and is inclined to the vertical at the angle

$$\tan^{-1}\frac{\sin\theta}{\theta\tan a}.$$

56. The pressure on the plane faces are double those in the previous question, as is also the pressure on the curved surface.

The direction of the resultant is therefore unchanged, while its magnitude is doubled.

57. Let $2d$ be the depth, A the area of the cylinder, $2W$ the weight of water it would contain.

Its weight $= W = wAd$.

Let x be the depth of the stop.

The pressure of the air inside exceeds that outside by

$$W/A = wd.$$

If h be the height of the water barometer, y the depth of the water below the stop,

$$\frac{h+x+y}{h} = \frac{d}{y} = \frac{h+d}{h};$$

$$\therefore\ x = d - \frac{hd}{h+d}.$$

If there be a hole in the stop and u be now its depth, v the length of the column of air in the cylinder,

$$\frac{h+u-d+v}{h} = \frac{2d}{v} = \frac{h+d}{h};$$

$$u = 2d - \frac{2hd}{h+d},$$

$$= 2x. \qquad \text{Q. E. D.}$$

58. At temperature zero let the portion not immersed occupy a length z of the tube.

Then $$z\left(1 + \frac{\tau}{6480}\right) = m.$$

If t' be the real temperature of the liquid, we should have (on wholly immersing the instrument),

$$t' = t - m + z\left(1 + \frac{t'}{6480}\right);$$

$$\therefore\ t' - t = -m + \left(1 + \frac{t'}{6480}\right)\frac{m}{1 + \frac{\tau}{6480}}$$

$$= m\left\{\frac{t' + 6480}{\tau + 6480} - 1\right\} = m\frac{t' - \tau}{\tau + 6480}$$

$$=m\frac{t'-t}{\tau+6480}+m\frac{t-\tau}{\tau+6480},$$

$$\therefore\ t'-t=\frac{m(t-\tau)}{6480+\tau-m}.$$

59. We have to find the weight of the liquid which would fill the space between the cone, vertical lines through all points of its base and a paraboloidal surface of semi-latus rectum g/ω^2 and vertex at the vertex of the cone.

If k be the height of such a surface

$$h^2\tan^2 a=\frac{2g}{\omega^2}k.$$

The required weight : that of the fluid in the cone as

$$\frac{2h}{3}+\frac{k}{2}:\frac{h}{3},$$

$$\therefore\ k=\tfrac{2}{3}h;$$

$$\therefore\ \omega=2\cot a\sqrt{\frac{g}{3h}}.$$

60. Let h be the height, $2a$ the vertical angle of the cone.

Horizontal pressure on the base $=w\pi h^3\tan^3 a$.

Vertical pressure on the curved surface

$$=\tfrac{1}{3}w\pi h^3\tan^2 a=W.$$

$\therefore$ the resultant $=W\sqrt{1+9\tan^2 a}$,

and makes an angle $\tan^{-1}3\tan a$ with the vertical.

It acts through the C.G. of the cone;

$\therefore$ (1) If its direction passes through C

$$\frac{h}{3}\Big/ h\tan a=3\tan a;$$

$$\therefore\ \tan a=\tfrac{1}{3}.$$

(2) If it is parallel to a generator

$$\cot a = 3 \tan a ;$$

$$\therefore \quad \tan a = \frac{1}{\sqrt{3}}, \quad \therefore \quad a = 30^\circ.$$

It can never be perpendicular to a generating line, for in that case it would make an angle a with the vertical. This could only be if $a=0$, i.e. the cone would become a straight line.

61. The areas of the two portions of the surface are as 3 : 1.

Let ρ, ρ' be the densities of the fluids.

The pressures at the depths of the C.G.'s of the two portions are as $\frac{4}{9}\rho \, : \, \rho + \frac{1}{3}\rho'$;

$\therefore$ the whole pressures are as

$$\tfrac{4}{3}\rho \, : \, \rho + \tfrac{1}{3}\rho'$$
$$= 4\rho \, : \, 3\rho + \rho'.$$

62. Let $x : y$ be the ratio in which the sides are divided. Then the pressures being equal at the two ends of the horizontal side,

$$x + 3y = y + 2x ;$$

$$\therefore \quad 2y = x,$$

or $$x : y = 2 : 1.$$

63. The depth of the vertex of the paraboloidal surface when the water has ceased to flow over is $\dfrac{r^2\omega^2}{2g}$.

The volume of water which has run over is $\therefore$

$$\tfrac{1}{2}\pi r^2 \cdot \frac{r^2\omega^2}{2g} = \tfrac{1}{4}\pi r^4 \omega^2/g$$

$$= \tfrac{3}{8}\frac{r\omega^2}{g} \times \text{volume of hemisphere.}$$

$\therefore$ the pressure on the table : original weight of liquid

$$= 1 - \tfrac{3}{8}\frac{r\omega^2}{g} : 1 = 8g - 3\omega^2 r : 8g.$$

64. The free surface would cut the cylinder at a distance $\frac{r^2\omega^2}{2g}$ above the top;

$\therefore$ the pressure on the top is

$$\frac{a}{\pi} \cdot \pi r^2 \cdot \frac{r^2\omega^2}{4g} = \frac{a}{4} \frac{r^4\omega^2}{g},$$

$2a$ being the angle of the wedge.

65. The whole pressure on the curved surface is that which would be produced on the same surface by fluid rising to a height $\frac{r^2\omega^2}{2g}$ above its actual height;

$\therefore$ h being the height, the pressure is

$$2arh \cdot \left(\frac{h}{2} + \frac{r^2\omega^2}{2g}\right) = arh^2 + ar^3\omega^2 h/g.$$

66. If we take c as the side of the isosceles triangle, and if we take c such that

$$\frac{2}{3} \frac{c}{\sqrt{2}} = \frac{a}{\sqrt{2}}, \text{ or } c = \frac{3a}{2},$$

the problem is the same as problem 9, page 207.

67. The pressure on the base remains constant, R (say).

If W be the weight of water displaced

$$(R\sqrt{1-s^2})^2 + (W + Rs)^2 = P^2;$$

$$\therefore \quad W^2 + R^2 + 2WRs = P^2.$$

Similarly $\quad W^2 + R^2 + 2WRs' = P'^2,$

$$W^2 + R^2 + 2WRs'' = P''^2;$$

$$\therefore \quad P^2(s' - s'') + P'^2(s'' - s) + P''^2(s - s') = 0.$$

68. If θ is the inclination of the axis to the vertical

$$\tan\theta = h\tan a \div \frac{h}{4} = 4\tan a.$$

The pressure on the base $= g\rho\pi h^3 \tan^3 a \sin\theta$.

That on the curved surface

$$= g\rho\pi h^2 \tan^2 a \operatorname{cosec} a \left(h \tan a \sin\theta + \frac{h}{3}\cos\theta\right).$$

The ratio of these is

$$\tan a \sin\theta : \operatorname{cosec} a (\tan a \sin\theta + \tfrac{1}{3}\cos\theta)$$
$$= 4\tan^2 a : \operatorname{cosec} a \{4\tan^2 a + \tfrac{1}{3}\}$$
$$= 12\sin^3 a : 12\sin^2 a + \cos^2 a$$
$$= 12\sin^3 a : 1 + 11\sin^2 a.$$

69. The horizontal pressure on the curved surface

$$= w\pi h^3 \tan^3 a \sin^2\theta.$$

The vertical pressure is

$$\tfrac{1}{3} w\pi h^3 \tan^2 a + w\pi h^3 \tan^3 a \sin\theta\cos\theta;$$

$$\therefore \quad \tan\phi = \frac{\tan a \sin^2\theta}{\frac{1}{3} + \tan a \sin\theta\cos\theta}$$
$$= \frac{16\tan^3 a}{\frac{1}{3}(1 + 16\tan^2 a) + 4\tan^2 a}$$
$$= \frac{48\tan^3 a}{1 + 28\tan^2 a},$$

or $$\cot\phi = \frac{28\cot a + \cot^3 a}{48}.$$

70. The only effect of the fluid is to reduce the apparent weight of the chain, for the resultant fluid pressure on each link is vertical. Thus the form remains the same as in air.

71. Let the surface divide the generating line in the ratio

$$x : 1 - x,$$

and let θ be the angle which the axis makes with the vertical, $2a$ the vertical angle of the cone;

$$\therefore \quad \cos 2a = \tfrac{2}{3}.$$

If l be the length of the side of the cone, the centre of the liquid surface is distant horizontally from the vertex

$$\tfrac{1}{2}\{xl\sin(\theta-a)+l\sin(\theta+a)\};$$

$\therefore$ the C.G. of the fluid is distant horizontally from the vertex

$$\frac{3l}{8}\{x\sin\overline{\theta-a}+\sin\overline{\theta+a}\}.$$

The point of suspension being vertically over this,

$$l\sin\overline{\theta-a}=\frac{3l}{8}\{x\sin\overline{\theta-a}+\sin\overline{\theta+a}\}.$$

But $$x=\frac{\cos(\theta+a)}{\cos(\theta-a)}=\frac{\sin 2\theta-\sin 2a}{\sin 2(\theta-a)};$$

$$\therefore\quad 8\sin(\theta-a)\cos(\theta-a)$$

$$=3\{\sin\overline{\theta-a}\cos\overline{\theta+a}+\sin\overline{\theta+a}\cos\overline{\theta-a}\},$$

$$4\sin 2(\theta-a)=3\sin 2\theta;$$

$$\therefore\quad \tan 2\theta=\frac{4\sin 2a}{4\cos 2a-3}=4\sqrt{5},$$

$$\sin 2\theta=\frac{4\sqrt{5}}{\sqrt{1+80}}=\frac{4\sqrt{5}}{9};$$

$$\therefore\quad \sin 2(\theta-a)=\frac{\sqrt{5}}{3}=\sin 2a;$$

$$\therefore\quad x=\tfrac{4}{3}-1=\tfrac{1}{3};$$

$$\therefore\quad x:1-x=1:2.$$

72. Let ρ, σ be the densities, θ the inclination, of the major axis to the vertical.

Then the pressures at the common surface being equal in the two fluids,

$$\rho(b\sin\theta+a\cos\theta)=\sigma(a\cos\theta-b\sin\theta);$$

$$\therefore\quad \tan\theta=\frac{\sigma-\rho}{\sigma+\rho}\cdot\frac{a}{b}.$$

73. Let h be the depth of the C.G. of the base.

The pressure on it is whA.

If θ be its inclination to the horizon, the pressure on the curved surface has for its

horizontal component $whA \sin \theta$,

vertical component $whA \cos \theta + wV$.

$\therefore$ the resultant pressure is $w[V^2 + 2hAV \cos \theta + h^2A^2]^{\frac{1}{2}}$,

$$P_1^2 = w^2[V^2 + 2x\,AV \cos \theta + x^2A^2],$$

$$P_2^2 = w^2[V^2 + 2y\,AV \cos \theta + y^2A^2],$$

$$P_3^2 = w^2[V^2 + 2z\,AV \cos \theta + z^2A^2];$$

$$\therefore \quad P_1^2(z-y) + P_2^2(x-z) + P_3^2(y-x)$$

$$= w^2A^2[x^2(z-y) + y^2(x-z) + z^2(y-x)]$$

$$= w^2A^2(x-y)(y-z)(z-x).$$

74. Considering the equilibrium of a small element s just beneath the surface, and of another just s' above it, let T be the tension at the surface, and let θ, θ' be the small angles between the tangents at the two ends of the elements. Then resolving, for each element, along the normal at the other end, we obtain, if a is the inclination to the horizontal of the string at the surface,

$$T \sin \theta = \rho\kappa s \cos(a + \theta),$$

$$T \sin \theta' = (\rho - \sigma)\kappa s \cos(a - \theta').$$

But, if r, r' are the radii of curvature,

$$s = r\theta \quad \text{and} \quad s' = r'\theta';$$

$\therefore$ making θ and θ' indefinitely small,

$$\frac{1}{r} : \frac{1}{r'} :: \rho : \rho - \sigma.$$

75. If W be the weight of the piston, A its area, and

$$W = A \cdot wy,$$

x being the depth of the fluid, if the piston be replaced by a depth y of fluid, the pressures throughout are the same, and $\therefore$ the jet rises to a height $x+y$.

76. Let ρ, σ be the densities of lead and water. The acceleration before reaching the water is g, when in the water it is $\frac{\rho-\sigma}{\rho}g$, for the resultant downward force on the ball is now its weight diminished by the upward pressure of the water, which is $\frac{\rho}{\sigma}$ times the weight.

$\therefore$ if V be the velocity on reaching the water, v the velocity at a depth x,

$$v^2 = V^2 + 2\frac{\rho-\sigma}{\rho}gx.$$

77. Let A be the volume of the receiver, B of the barrel, t the interval between two successive strokes.

After r strokes the density of the air is

$$\left(\frac{A}{A+B}\right)^r \rho,$$

ρ being its initial density.

$\therefore$ the acceleration now is $\left\{1-\left(\frac{A}{A+B}\right)^r\right\}g$.

The velocity at the nth stroke is

$$gt\left[n-1-\frac{A}{A+B}-\left(\frac{A}{A+B}\right)^2-\dots-\left(\frac{A}{A+B}\right)^{n-1}\right]$$

$$=gt\left\{n-\frac{1-\left(\frac{A}{A+B}\right)^n}{1-\left(\frac{A}{A+B}\right)}\right\}$$

$$=gt\left\{n-\frac{(A+B)^n-A^n}{B(A+B)^{n-1}}\right\}=gt\left\{n-1-\frac{A}{B}\left[1-\left(\frac{A}{A+B}\right)^{n-1}\right]\right\}.$$

Just before the $\overline{n+1}$th stroke the velocity is

$$gt\left\{n-\frac{A}{B}\left[1-\left(\frac{A}{A+B}\right)^n\right]\right\}.$$

78. $v=$the limit when n is infinite of

$$gnt\left\{\frac{n}{n+1}\right\}^n$$

$$=\text{the limit of } gnt\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{gnt}{e}=\frac{v'}{e},$$

or $$v'=ev.$$

79. If r, r' be its radii at depths a, $\frac{1}{2}a$, p, p' the internal pressures, then $p : p' :: r'^3 : r^3$.

If we neglect atmospheric pressure and also surface tension,

$$r'^3=2r^3;$$

but if we take account of surface tension,

$$p : p'=r'^3 : r^3,$$

$$p-wa=\frac{2t}{r},$$

$$p'-w\frac{a}{2}=\frac{2t}{r'}=\frac{r}{r'}(p-wa),$$

$$p\frac{r^3}{r'^3}-\frac{r}{r'}(p-wa)-w\frac{a}{2}=0,$$

which equation determines the ratio of the radii in the two cases.

80. Since the volume of the paraboloid is one-half that of a cylinder of same height and base, the depth of the vertex of the paraboloid when the water reaches the rim is $\frac{2h}{n}$.

$$\therefore \quad a^2 = \frac{2g}{\omega^2} \cdot \frac{2h}{n},$$

or
$$\omega = \frac{2}{a}\sqrt{\frac{gh}{n}}.$$

81. Let x be the length of the fluid in the leg AB, $\therefore$ $l-x$ the length in BC, α the angle ABC.

Since the free surface must pass through both ends of the fluid, its vertex being at the end in AB,

$$(l-x)^2 \sin^2\alpha = \frac{2g}{\omega^2}\{(l-x)\cos\alpha - x\}$$

is the equation giving x in terms of l.

We obtain

$$l-x = \frac{g}{2\omega^2 \sin^2 \frac{\alpha}{2}}\left\{1 \pm \sqrt{1 - \frac{2\omega^2 l}{g}\tan^2\frac{\alpha}{2}}\right\}.$$

If
$$\omega^2 > \frac{g}{2l}\cot^2\frac{\alpha}{2},$$

this expression becomes imaginary.

Now the greatest value which $l-x$ can have is

$2x \sec \alpha$. (See Art. 164 (2)).

$\therefore$ if
$$l > x(1 + 2\sec\alpha),$$
some of the fluid will run out.

In the limiting case,

$$4x^2 \tan^2\alpha = \frac{2g}{\omega^2} \cdot x;$$

$$\therefore \quad x = \frac{g}{2\omega^2}\cot^2\alpha,$$

$$l = \frac{g}{2\omega^2}\cot\alpha\,[\cot\alpha + 2\,\mathrm{cosec}\,\alpha],$$

$$\frac{2\omega^2 l}{g} = \frac{\cos^2 a + 2\cos a}{\sin^2 a} = \cot^2\frac{a}{2} - \operatorname{cosec}^2 a.$$

∴ before ω reaches the value given, water begins to flow out of the tube and continues so to flow till all is gone.

82. Let $2a$ be the angle between the rods, σ, ρ the density of the rods and water, θ the small angular displacement,

$$a\sigma = c\rho.$$

The length of one rod immersed increases to

$$c\cos a \sec(a+\theta) = c(1+\theta\tan a).$$

The moment of fluid pressure on it round A is

$$\tfrac{1}{2}g\rho\,.\,c^2(1+\theta\tan a)^2\sin(a+\theta)$$

$$=\tfrac{1}{2}g\rho c^2\{\sin a + \theta(\sec a + \tan a \sin a)\}.$$

The moment of its weight is $\frac{1}{2}\,.\,g\sigma a\,.\,a\sin(a+\theta)$

$$=\tfrac{1}{2}g\rho ac(\sin a + \theta\cos a),$$

∴ the equilibrium is stable if

$$g\rho c^2\,.\,\theta(\sec a + \tan a\sin a) > g\rho ac\,.\,\theta\cos a,$$

i.e. if $$c(\sec a + \tan a \sin a) > a\cos a,$$

i.e. if $$c\left\{1+\frac{1-\cos 2a}{2}\right\} > a\,\frac{1+\cos 2a}{2},$$

i.e. if $$c(3-\cos\omega) > a(1+\cos\omega);$$

$$\because\quad \omega = 2a.$$

83. Any such area as $A_nA_1A_2A_3$ must be a maximum consistently with keeping its sides constant.

∴ it must be inscribable in a circle, and hence the whole polygon is inscribable in a circle.

If R be the radius of this circle,

$$R = \frac{c_1}{\sin a_1} = \frac{c_2}{\sin a_2} = \frac{c_3}{\sin a_3} = \dots\dots$$

84. If h be the height of a cylinder, a its radius, ρ the density of the water in it, w the weight of that water,

$$w = \rho g \, . \, \pi a^2 \, . \, h.$$

The pressure on the curved surface

$$= 2\pi a \, . \, h \, . \, \rho g \, . \, \frac{h}{2} = \rho g \pi a h^2$$

$$= \frac{h}{a} w.$$

If the downward acceleration be f, the pressures are all diminished in the ratio $g : g - f$.

If the acceleration f be upwards they are increased in the ratio $g : g + f$.

$\therefore$ the pressures are

$$\frac{g+f}{g} w \, . \, \frac{h}{a},$$

$$\frac{g-f}{g} w' \frac{h'}{a'}.$$

And $$f = \frac{w' - w}{w' + w} g.$$

$$\therefore \quad \frac{g-f}{g} = \frac{2w}{w' + w},$$

$$\frac{g+f}{g} = \frac{2w'}{w' + w};$$

$\therefore$ the pressures are as $\frac{h}{a} : \frac{h'}{a'}$.

If the cylinders be similar the pressures are equal, if they be of equal height, the pressures are in the inverse ratio of the radii.

85. Taking the case of a hole at an angular distance θ from the highest point of the circle.

Its depth is $a(1 - \cos\theta)$.

$\therefore$ the time of falling to the level of the lowest point is

$$\sqrt{\frac{2a(1+\cos\theta)}{g}}.$$

$\therefore$ the distance from the plane face of the point where the fluid from this hole meets the horizontal plane through the lowest point is

$$\sqrt{2ga(1-\cos\theta)}\,.\,\sqrt{\frac{2a}{g}(1+\cos\theta)}=2a\sin\theta.$$

The horizontal distance from the centre is $a\sin\theta$.

$\therefore$ the point lies on a line through the lowest point inclined to the vertical face at an angle $\tan^{-1}2$, i.e. the trace on the plane is two straight lines.

86. The air in the jar can vibrate freely in the same period as the tuning-fork in the first case, but not in the other cases. It is therefore set in vibration, giving out the note natural to a jar of that depth in the first case, but being unable to vibrate in the same period as the fork in the other cases, gives but a slight sound.

87. The sound of the clapping is reflected from each rail and the series following rapidly one on the other and falling on the ear produces a sound resembling that produced by a cause which sets the air in vibration along a definite line, not instantaneously, but in rapid succession.

88. If W is given weight, W' weight of cone, and w of water,

$$\text{acceleration}=f=\frac{W-W'-w}{W+W'+w}g.$$

Hence $\frac{w}{g}\,.\,f=$ upward force on the water,

$$=\text{resultant vertical pressure}-w,$$

and whole pressure $=\operatorname{cosec}\alpha$ (resultant pressure).

89. If ϖ_1 ϖ_2 be the new pressures, τ the increase of temperature,

$$\frac{\varpi_1}{273+t+\tau}=\frac{\Pi}{273+t},$$

$$\frac{\varpi_2}{273+t'+\tau}=\frac{\Pi}{273+t'};$$

$$\therefore \quad \varpi_1=\Pi\left(1+\frac{\tau}{273+t}\right),$$

$$\varpi_2=\Pi\left(1+\frac{\tau}{273+t'}\right).$$

$\therefore$ the increase of pressure is greater in that which had the lower temperature originally.

The pressure at zero in one will be

$$\frac{273\,\Pi}{273+t}=\Pi\left(1-\frac{t}{273}\right) \text{ approximately.}$$

That in the other will be $\Pi\left(1-\frac{t'}{273}\right)$.

When the air in both is forced into the same vessel, the pressure will be the sum of these two, i.e. $\Pi\left(2-\frac{t+t'}{273}\right)$.

90. Let Π, ϖ be the two pressures and $a=\frac{1}{273}$,

$$\frac{\Pi}{273}=\frac{\varpi n^3}{273+t};$$

$$\therefore \quad \Pi : \varpi = n^3 : 1+at.$$

91. The mercury expands till it fills

$$\tfrac{1}{2}(1\cdot0036)=\cdot5018 \text{ of the vessel.}$$

$\therefore$ the air now fills $\cdot4982$ of the vessel.

If Π, ϖ be the pressures of the air at 0° and 20°,

$$\frac{\Pi\times\cdot5}{273}=\frac{\varpi\times\cdot4982}{293};$$

$$\therefore \quad \frac{\varpi}{\Pi}=\frac{293}{273\times\cdot9964}=1\cdot07716...$$

92. If ρ be the density of the body, σ that of the air,

$$\frac{1}{m} = \frac{\sigma}{\rho}.$$

When the barometer reading changes from h to h', the density of the air becomes $\frac{h'}{h}\sigma$.

It now loses $\frac{h'}{h}\frac{\sigma}{\rho} = \frac{h'}{mh}$ of its weight.

93. Let V be the volume of a given mass of mercury at 68°.

At 212° its volume is $\frac{70}{69}V$.

The volume of an equal mass of water at 68° is

$$13{\cdot}568\,V.$$

At 212° it is $13{\cdot}704 \times \frac{70}{69}V$.

$\therefore$ the proportional expansion of water is

$$\frac{13{\cdot}704 \times 70 - 13{\cdot}568 \times 69}{13{\cdot}568 \times 69} = \frac{23088}{936192},$$

$$= {\cdot}02466\ldots = \tfrac{2}{81} \text{ nearly}.$$

94. The area of the surface as far as the rth plane

$$= \frac{h_r}{a} \times \text{whole surface}.$$

$\therefore$ the whole pressure on this part

$$= \frac{h_r^2}{a^2} \times \text{pressure on whole}.$$

$$\therefore \quad \frac{h_r^2}{a^2} = \frac{r}{n},$$

or

$$\frac{h_r}{a} = \sqrt{\frac{r}{n}}.$$

95. If the hexagon be divided into six triangles by joining the angular points to the centre, and if a be the depth of the centre, the depths of the centres of gravity are

$$\frac{a}{3}, \frac{2a}{3}, \frac{4a}{3}, \frac{5a}{3}.$$

$\therefore$ the whole pressures are in these ratios.

The centres of pressure are at depths

$$\frac{a}{2}, \frac{3a}{4}, \frac{11}{8}a, \frac{17}{10}a. \quad \text{(See p. 207.)}$$

The depth of the centre of pressure of the whole hexagon is

$$a \cdot \frac{\frac{1}{2}+2.2.\frac{3}{4}+2.4.\frac{11}{8}+5.\frac{17}{10}}{1+2.2+2.4+5} = \frac{23}{18}a.$$

96. The triangle may be divided into a number of small triangles with equal bases and the same vertex.

The centre of pressure of each of these is $\frac{3}{4}$ its length from the vertex.

$\therefore$ the centre of pressure of the whole is in the generator of the cylinder which passes through the vertex of the triangle, and divides it in the ratio 3 : 1.

97. Let v be the volume of each sphere,

ρ, ρ' their densities, σ that of the fluid.

If possible let equilibrium exist when the spheres are at distances a, b from the axis of rotation, and θ is the angle which the string makes with that axis.

Then,
$$T\sin\theta + \sigma v \,.\, \omega^2 a = \rho v \omega^2 a,$$
$$-T\sin\theta + \sigma v \,.\, \omega^2 b = \rho' v \omega^2 b,$$
$$T\cos\theta + g\sigma v = g\rho v,$$
$$-T\cos\theta + g\sigma v = g\rho' v.$$

Eliminating T, we obtain the equations,

$$2\sigma=\rho+\rho', \qquad \sigma(a+b)=\rho a+\rho' b,$$

and
$$g(a+b)\tan\theta=2ab\omega^2;$$

also, if l is the length of the string, $l\sin\theta=a-b$; we have therefore three equations of condition.

98. Let 2θ be the angle between the planes, 2α the vertical angle of the cone, and h its height.

The pressure on each plane face is $\frac{1}{6}wh^3\tan\alpha$.

$\therefore$ the resultant horizontal pressure on the curved surface $=\frac{1}{3}wh^3\tan\alpha\sin\theta$.

The resultant vertical pressure

$$=\tfrac{1}{3}w\theta h^3\tan^2\alpha.$$

$\therefore$ the resultant pressure

$$=\tfrac{1}{3}wh^3\tan\alpha\sqrt{\sin^2\theta+\theta^2\tan^2\alpha}.$$

The C.G. of the surface is at a distance from the centre

$$\frac{2}{3}\cdot\frac{h\tan\alpha\sin\theta}{\theta}.$$

$\therefore$ the C.G. of the contained fluid (through which the resultant pressure acts) is distant $\frac{1}{2}\,\frac{h\tan\alpha\sin\theta}{\theta}$ from the axis, and $\therefore$ the line joining it to the centre of the base of the cone makes an angle $\tan^{-1}\frac{2\tan\alpha\sin\theta}{\theta}$ with the vertical.

The resultant pressure makes an angle

$$\tan^{-1}\frac{\sin\theta}{\theta\tan\alpha} \text{ with the vertical.}$$

$\therefore$ its line of action passes through the centre of the base if

$$\tan^2\alpha=\tfrac{1}{2}, \text{ i.e. if } \alpha=45^\circ.$$

99. Considering the triangle of forces for each sphere it follows that the vertical through the point, O, bisects the distance between the spheres.

If u is the distance between O and the middle point,

$$2u^2+\frac{x^2}{2}=r^2+r'^2.$$

If θ is the inclination of the line joining the spheres to the horizontal,

$$r^2=u^2+\frac{x^2}{4}\mp ux\sin\theta, \qquad r'^2=u^2+\frac{x^2}{4}\pm ux\sin\theta;$$

$$\therefore \quad \sin\theta=\frac{r^2\sim r'^2}{2ux}=\frac{r^2\sim r'^2}{x\sqrt{2(r^2+r'^2)-x^2}}.$$

Also if P is the excess of fluid pressure over the weight of a sphere

$$\phi(x) : P :: \frac{x}{2} : u.$$

100. A being the area of the piston, Π the atmospheric pressure, x the length of the string in equilibrium.

The pressure on the piston is $\frac{a}{x}\Pi . A$.

$$\therefore \quad x=l\left[1+\frac{\frac{a-x}{x}\Pi A}{\Pi A}\right]$$

$$=\frac{la}{x};$$

$$\therefore \quad x=\sqrt{la}.$$

101. Let a be the length of the barometer tube.

The volume of the air under a pressure $\cdot 4$ inches is $a-29\cdot 4$.

Under pressure $\cdot 5$ inches it is $a-29\cdot 9$.

$$\therefore \quad 5(a-29\cdot 9)=4(a-29\cdot 4);$$

$$\therefore \quad a=31\cdot 9 \text{ inches},$$

i.e. when the reading is 29, the air occupies $2\cdot 9$ inches.

If the true reading be x, the pressure of the air is

$$x-29.$$

$$\therefore \quad (x-29)\,2{\cdot}9 = {\cdot}5\,(a-29{\cdot}9) = 1{\cdot}0,$$

$$29x = 851,$$

$$x = 29{\cdot}3448\ldots$$

102. If $2h$ be the length of the cylinder, a length h will be in the upper, a length h in the lower liquid.

The pressure on the upper end is due to a depth h of a liquid of density ρ.

That on the lower end is due to a depth $2h$ of liquid of density ρ, and a depth h of liquid of density 3ρ, i.e. is equal to that produced by a depth $5h$ of liquid of density ρ, and is therefore five times that on the upper end.

103. The pressure on the lower end, being equal to the weight of water displaced by the rope, is equal to the weight of one-half the immersed rope.

The tension at the middle section of the immersed rope is therefore zero, since the weight of the portion below that section is just supported by the fluid pressure on the lower end.

104. If z be the depth of the centre of pressure

$$2z\,(a+b+c) = a^2+b^2+c^2+bc+ca+ab.$$

The depth of the centre of gravity is $\dfrac{a+b+c}{3}$.

Now

$$z-\frac{a+b+c}{3} = \frac{a^2+b^2+c^2+bc+ca+ab}{2\,(a+b+c)} - \frac{a+b+c}{3}$$

$$= \frac{a^2+b^2+c^2-bc-ca-ab}{6\,(a+b+c)}$$

$$= \frac{(b-c)^2+(c-a)^2+(a-b)^2}{12\,(a+b+c)}.$$

105. If one point of the disc of radius R were in the surface, the centre of pressure would be at a distance $\frac{p}{r}R$ from the centre.

The pressure would be $g\rho\,.\,\pi R^3$.

If liquid be now added till the surface is raised to a distance h above the centre, the pressure is increased to $g\rho\pi R^2\,.\,h$, the moment about the horizontal line through the centre being unaltered, since the resultant of the addition pressures acts through the centre.

If q be now the distance of the centre of pressure from the centre,

$$g\rho\pi R^2\,.\,h\,.\,q = g\rho\pi R^3\,.\,\frac{p}{r}R,$$

or

$$q = pR^2 \div hr.$$

106. If the mercury rise through a distance x, the air which originally filled a space $A+\kappa l$ expands so as to fill a space

$$A+\kappa(l-x)+B.$$

Its pressure becomes $\frac{h-x}{h}$ of its original value.

$$\therefore\quad h(A+\kappa l) = (h-x)\{A+B+\kappa l-\kappa x\}.$$

If we neglect κx^2 we have

$$x = \frac{hB}{A+B+\kappa(l+h)} = \frac{hB}{A+B}\left[1-\frac{\kappa(l+h)}{A+B}\right] \text{ q. p.}$$

Writing this value for x in the neglected term and inserting it,

$$x = \frac{hB}{A+B}\left[1-\kappa\frac{l+h}{A+B}\right]+\kappa\frac{h^2B^2}{(A+B)^3}$$

$$= \frac{hB}{A+B}\left[1-\kappa\frac{Ah+(A+B)\,l}{(A+B)^2}\right].$$

107. Let θ be the angle made by the base of the hemisphere with the vertical, when the attached weight on the rim is just in the surface of the water.

The volume of water displaced is $\frac{(1-\cos\theta)^2(2+\cos\theta)}{2}$ × volume of hemisphere.

If w, W be the weights of the hemisphere and of the water which would fill it

$$\tfrac{5}{4}w=\tfrac{1}{2}\,.\,W(1-\cos\theta)^2(2+\cos\theta).$$

Also taking moments about the centre (through which the fluid pressure passes)

$$\frac{w}{4}\,.\,a\sin\theta=w\,.\,\frac{a}{2}\cos\theta\,;$$

$$\therefore\quad \tan\theta=2,$$

$$\therefore\quad \cos\theta=\frac{1}{\sqrt{5}},$$

$$\therefore\quad (1-\cos\theta)^2(2+\cos\theta)=\frac{10\sqrt{5}-14}{5\sqrt{5}},$$

$$\therefore\quad W:w=25\sqrt{5}:20\sqrt{5}-28.$$

108. Let the external atmospheric pressure increase till the water-barometer reading is $H+u$.

Let d be the depth of the top of the bell originally.

The internal pressure is $\frac{h}{x}H$ and also $H+d+x$.

$$\therefore\quad hH=x(H+d+x).$$

If the bell be free to move, it will continue to displace the same volume of water, i.e. x remains unaltered.

Let y be the amount of its motion,

$$h(H+u)=x[H+u+d+y+x]$$
$$=hH+x(u+y),$$
$$y=u\,.\,\frac{h-x}{x}.$$

If the bell be held fixed, let z be the fall of the water in it. Then

$$h(H+u)=(x+z)[H+u+d+x+z]$$
$$=hH+x(u+z)+z[H+d+x+u-z].$$

Neglecting squares of small quantities

$$z=u\cdot\frac{h-x}{H+d+2x};$$

$$\therefore\quad y:z=H+d+2x:x$$
$$=x+\frac{hH}{x}:x$$
$$=Hh+x^2:x^2.$$

109. If h, h' be the heights of the cones,

α, β the areas of their bases,

the weights of the contained water are as $\alpha h:\beta h'$.

$\therefore$ the acceleration is $\dfrac{\alpha h-\beta h'}{\alpha h+\beta h'}g=f$.

The pressure on the bases are as

$$\frac{g-f}{g}\alpha h:\frac{g+f}{g}\beta h',$$

which are always equal.

The weights in the second case are

$$\alpha h(1-m^3),\qquad \beta h'(1-n^3).$$

The pressures on the bases are as

$$(g-f)\,\alpha h(1-m):(g+f)\,\beta h'(1-n),$$

where $$f=\frac{\alpha h(1-m^3)-\beta h'(1-n^3)}{\alpha h(1-m^3)+\beta h'(1-n^3)};$$

$\therefore$ the pressures are as

$$(1-n^3)(1-m):(1-m^3)(1-n),$$

i.e. as $$1+n+n^2:1+m+m^2.$$

110. If the required point divide the side of the cone in the ratio $x : 1-x$, h being the height, $2a$ the angle of the cone,

The water issues with a vertical velocity upwards $\sqrt{2ghx}\sin a$ and a horizontal velocity $\sqrt{2ghx}\cos a$.

If it is to fall just outside the base, the time of flight must be

$$\frac{(1-x)\,h\tan a}{\sqrt{2ghx}\cos a},$$

$\therefore$ $(1-x)\,h=$ the vertical space described in this time,

$$=\tfrac{1}{2}g\,\frac{(1-x)^2\,h^2\tan^2 a}{2ghx\cos^2 a}-\sqrt{2ghx}\sin a\,.\,\frac{(1-x)\,h\tan a}{\sqrt{2ghx}\cos a};$$

$$\therefore\quad 1=\frac{(1-x)\tan^2 a}{4x\cos^2 a}-\tan^2 a,$$

or

$$(1-x)\tan^2 a=4x.$$

$$\therefore\quad x=\frac{\tan^2 a}{4+\tan^2 a}.$$

111. The time of vibration of the fifth of G is $\frac{2}{3}$ that of G and therefore $\frac{4}{9}$ that of C.

112. Let ρ, σ be the densities of the fluid and pyramid.

The depth immersed being k

$$\rho k^3=\sigma h^3.$$

The horizontal pressure of the fluid perpendicular to the dividing plane is $g\rho\,.\,k\,.\,\frac{k}{h}a\,.\,\frac{k}{3}$.

The centre of pressure is at a distance $\frac{k}{2}$ from the hinge.

$\therefore$ the moment of horizontal fluid pressure about the hinge is

$$\tfrac{1}{6}g\sigma ah^2k.$$

The vertical fluid pressure on the half pyramid = the weight of the half-pyramid $= \frac{2}{3} g\sigma a^2 h$.

The centre of gravity of the half-pyramid is $\frac{3}{8}a$ from the dividing plane.

The line of action of the fluid pressure is $\frac{3}{8}\frac{k}{h}a$ from that plane.

$\therefore$ in order that the parts may remain in contact

$$\frac{1}{6} g\sigma a h^2 k > \frac{2}{3} g\sigma a^2 h \, . \, \frac{3}{8} a \left(1 - \frac{k}{h}\right),$$

$$\text{i.e. } 2h^2 k > 3a^2 (h-k),$$

$$k(2h^2 + 3a^2) > h \, . \, 3a^2.$$

$$\text{i.e. } \frac{\sigma}{\rho} > \left(\frac{3a^2}{3a^2 + 2h^2}\right)^3.$$

113. The velocity of the particle on reaching the fluid is $\sqrt{2ga \cos \alpha}$, a being the radius of the axis of the tube.

Let m, M be the masses of the particle and fluid, which we shall suppose inelastic, u the velocity acquired by the fluid.

Then $$(m+M)\, u = m \sqrt{2ga \cos \alpha},$$

$$\therefore \quad u = \frac{m}{m+M} \sqrt{2ga \cos \alpha}.$$

At a distance θ from the particle, the impulsive pressure is such as to produce the velocity u in the portion beyond that point;

$$\therefore \text{ the impulse } = \frac{2a - \theta}{2a} Mu$$

$$= \left(1 - \frac{\theta}{2a}\right) \frac{Mm}{M+m} \sqrt{2ga \cos \alpha}.$$

114. Let h be the height, r the radius of the cylinder.

The impulsive pressure at a depth x is such as to

destroy a velocity v in a column of fluid of height x and $\therefore = \rho x \,.\, v$.

The whole impulse on the curved surface

$$= v\rho \,.\, 2\pi r h \,.\, \frac{h}{2} = v\rho\pi r h^2.$$

115. The impulse at the depth x below the vertex $= \rho v x$.

The resultant impulse on the base $= \rho v h \,.\, \pi h^2 \tan^2 a$, and the whole impulse on the curved surface

$$= \rho v \,.\, \pi h^2 \tan^2 a \operatorname{cosec} a \,.\, \tfrac{2}{3} h \tan a.$$

116. Let v be the velocity with which the sphere reaches the plane.

Using the same principle as in the last two examples,

The whole impulse $= \rho v r \,.\, 4\pi r^2 = 4\pi\rho v r^3$.

Impulse on upper half $= \rho v \,.\, \frac{r}{2} \,.\, 2\pi r^2 = \pi\rho v r^3$.

Impulse on lower half $= \rho v \frac{3r}{2} \,.\, 2\pi r^2 = 3\pi\rho v r^3$.

The resultant impulse on the whole surface

$$= \tfrac{4}{3} \,.\, \rho v \pi r^3.$$

Let P be the resultant impulse on the horizontal diametral section, Q, R those on the upper and lower hemispheres.

Then
$$P - Q = \tfrac{2}{3}\rho v \pi r^3,$$
$$R - P = \tfrac{2}{3}\rho v \pi r^3.$$

And
$$P = \rho v \,.\, \pi r^2 \,.\, r = \rho v \pi r^3,$$
$$\therefore \quad Q = \tfrac{1}{3}\rho v \pi r^3, \qquad R = \tfrac{5}{3}\rho v \pi r^3.$$

117. Let p be the pressure of the air forced in,

t the tension in the material of the tube,

$\frac{t}{p}$ is the radius of the portions not in contact with the sides of the triangle.

$\therefore \quad \frac{2\pi t}{p}$ is their length.

If a be the length of the side of the triangle, the portions in contact with each side are $a - 2\sqrt{3}\,\frac{t}{p}$.

The original circumference of the tube was $\frac{\pi a}{\sqrt{3}}$.

$$\therefore \quad 3a + t\left[\frac{2\pi}{p} - \frac{6\sqrt{3}}{p}\right] = \frac{\pi a}{\sqrt{3}}\left[1 + \frac{t}{\lambda}\right],$$

whence t is determined and thus the other quantities required.

118. The resultant of the tensions round a horizontal section is equal to the resultant vertical pressure on the portion of the cone above that section, i.e. between it and the vertex.

If h be the height of the cone, x the depth of the section, t the tension at the section,

$$2\pi x \tan\alpha \,.\, t \,.\, \cos\alpha = \tfrac{2}{3}\rho g \pi x^3 \tan^2\alpha,$$

$$\therefore \quad t = \tfrac{1}{3}\rho g \tan\alpha \sec\alpha \,.\, x^2.$$

119. Let h be the height of the bag, $4a$ the parameter of the parabola, so that if r be the radius at a distance x from the vertex $r^2 = 4ax$.

t being the tension,

$$2\pi r \,.\, t \,.\, \frac{2x}{\sqrt{r^2 + 4x^2}} = g\rho\pi r^2 \,.\, \left(h - \frac{x}{2}\right);$$

$$\therefore \quad t = \tfrac{1}{4} g\rho r \,.\, \sqrt{\frac{(x+a)}{x}}\,(2h - x)$$

$$= \tfrac{1}{2} g\rho \sqrt{a(x+a)}\,(2h - x).$$

At the vertex $x = 0$,

$$\therefore \quad t = g\rho a h.$$

The radius of curvature at the vertex is $2a$.

The pressure there is $g\rho h$.

$$\therefore \quad \frac{2t}{2a}=g\rho h,$$

or

$$t=g\rho a h.$$

120. $$t\,.\,2\pi\,.\,PN\,.\,\frac{2AN}{\sqrt{PN^2+4AN^2}}=g\rho\pi PN^2\,.\,\frac{AN}{2},$$

$$\therefore \quad t=g\rho\sqrt{\frac{AS+AN}{AS}}\,.\,\frac{AS\,.\,AN}{2}$$

$$=\tfrac{1}{2}g\rho\sqrt{AS}\,.\,\sqrt{SP}\,.\,AN,$$

$$\propto AN\,.\,\sqrt{SP}.$$

121. Let l be the length of the faulty barometer,

γ the true reading when its reading is c.

A length $l-a$ of the tube is filled with air at a pressure $a-a$.

A length $l-b$ of the tube is filled with air at a pressure $\beta-b$.

A length $l-c$ of the tube is filled with air at a pressure $\gamma-c$.

$$\therefore \quad (l-a)(a-a)=(l-b)(\beta-b)=(l-c)(\gamma-c),$$

$$l(a-\beta-a+b)=a(a-a)-b(\beta-b).$$

$$\therefore \quad l-a=\frac{(a-b)(\beta-b)}{a-a-(\beta-b)},$$

$$l-c=\frac{(a-c)(a-a)-(b-c)(\beta-b)}{a-a-(\beta-b)},$$

and

$$\gamma-c=\frac{(l-a)(a-a)}{l-c}$$

$$=\frac{(a-a)(\beta-b)(a-b)}{(a-c)(a-a)-(b-c)(\beta-b)}.$$

122. Let $2\theta_r$ be the angle made by the rth bounding radius with the surface, and let there be n sectors, a being the radius.

The area of the first r sectors is $a^2\theta_r$.

The depth of the centre of gravity is $a\dfrac{\sin^2\theta_r}{\theta_r}$.

$$\therefore \quad g\rho \,.\, a^3 \sin^2\theta_r = \frac{r}{n} g\rho \,.\, \frac{\pi a^2}{2} \,.\, \frac{2a}{\pi},$$

$$\sin^2\theta_r = \frac{r}{n},$$

or the rth radius makes an angle $2\sin^{-1}\sqrt{\dfrac{r}{n}}$ with the surface.

123. Let h be the height of the cylinder, a its area.

The pressure on the conical surface when it is uppermost is

$$\tfrac{2}{3}a \,.\, 3h \,.\, g\rho = 2g\rho a h.$$

When it is lowest it is

$$g\rho a h + \tfrac{1}{3} g\rho \,.\, a \,.\, 3h = 2g\rho a h.$$

124. The weight of the air in the balloon is

$$\frac{1000}{1728} \times 24 \times 20 \times \frac{1}{800} \text{ grains} = \frac{25}{72} \text{ grains}.$$

At a depth x feet it displaces $\dfrac{33}{33+x}$ cubic inches of water the weight of which is $\dfrac{33}{33+x} \times \dfrac{2500}{9}$ grains.

The weight of the lead in water is

$$\left(1 - \frac{80}{912}\right) 100 = \frac{5200}{57} \text{ grains}.$$

$$\therefore \quad \frac{5200}{57} + \frac{25}{72} = \frac{33}{33+x} \,.\, \frac{2500}{9}$$

$$\therefore \quad x = 66{\cdot}44\ldots \text{ feet}.$$

125. If v is the volume of the hydrometer at first, and v' after expansion, and if ρ and ρ' are the densities of the fluid,

$$\rho(v-\kappa x)=\rho'(v-\kappa x_1)=\rho'(v'-\kappa x_2),$$

$$\therefore \quad v'-v=\kappa(x_2-x_1) \quad \text{and} \quad \left(\frac{\rho}{\rho'}-1\right)v=\kappa(x-x_1).$$

$$\therefore \quad \frac{v'-v}{v} : \frac{1/\rho'-1/\rho}{1/\rho} :: x_2-x_1 : x-x_1.$$

126. The resultant pressure of the liquid is in the vertical through the centre of the hemisphere, and, taking moments about the centre

$$\tfrac{3}{8}a\sin\theta \,.\, W=c\cos\theta \,.\, w.$$

127. Let U and V be the volumes of the hemisphere and the cylinder, W the weight of the float.

Then taking the hemisphere as the lowest portion,

$$W=(U+V)s=V\,.\,3s=\left(U+\frac{x}{h}V\right)2\rho.$$

$$\therefore \quad x:h::1:4.$$

128. Let $ABCDE$ be the pentagon, A being the lowest vertex, and AF the perpendicular from A upon CD.

Then if c is the side of the hexagon,

$$\frac{a}{2}=AF=c\cos 54^\circ+c\cos 18^\circ=2c\cos 36^\circ\,.\,\cos 18^\circ.$$

The depth of A below the surface being a, the

depth of C and D is $\dfrac{a}{2}$, and

depth of $B=\dfrac{a}{2}+c\sin 72^\circ=a\dfrac{\sqrt{5}+1}{4}$.

Now it is shewn on page 207 that, if α, β, γ are the depths of the angular points of a triangle, and z the depth of its centre of pressure,

$$2z(\alpha+\beta+\gamma)=\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta.$$

Employing this formula we find that the depths of the centres of pressure of ACD and ABC are respectively

$$\frac{11}{16}a \text{ and } \frac{5+\sqrt{5}}{7+\sqrt{5}}a.$$

Let α represent the area ABC;

then area ACD

$$=\alpha \cdot \frac{AF \cdot CF}{\frac{1}{2}c^2 \sin 108^\circ} = \alpha \frac{a^2}{2c^2} \frac{\tan 18^\circ}{\sin 72^\circ} = \alpha \frac{\sqrt{5}+1}{2}.$$

The depths of the centroids of ACD and ABC are respectively $\frac{2}{3}a$, and $\frac{a}{12}(7+\sqrt{5})$.

Hence if $\bar{z}$ is the depth of the centre of pressure of the pentagon,

$$\bar{z}\left\{\frac{\sqrt{5}+1}{3}a\alpha+\frac{7+\sqrt{5}}{6}a\alpha\right\}$$

$$=\frac{\sqrt{5}+1}{3}a\alpha \cdot \frac{11a}{16} + \frac{7+\sqrt{5}}{6}a\alpha \cdot \frac{5+\sqrt{5}}{7+\sqrt{5}}a,$$

and $\therefore$ $$\bar{z}=\frac{a}{48}(29+3\sqrt{5}).$$

129. If z be the depth of the surface of the water in the bell,

α the area, p the pressure,

the work done in depressing the water surface through a small distance x is

$$p\alpha \cdot x$$

$$=p \times \text{volume of water displaced,}$$

$$=g\rho z \times \text{volume of a slice of thickness } x \text{ of the bell at depth } z.$$

The whole work done in displacing the water is the sum of such portions and since

Σ ($z \times$ volume of slice at depth z)

$=$ Whole volume displaced $\times$ depth of its centre of gravity,

The work done
= Weight of water displaced × depth of its centre of gravity.

130. Let h be the breadth of the quadrilateral.

The centre of pressure of ACD is at a depth $\frac{3}{4}h$.

The centre of pressure of ABC is at a depth $\frac{1}{2}h$.

The pressure on ACD is $\frac{1}{2}wh \,.\, CD \,.\, \frac{2h}{3} = \frac{1}{3}wh^2 \,.\, CD$.

The pressure on ABC is $\frac{1}{2}wh \,.\, AB \,.\, \frac{h}{3} = \frac{1}{6}wh^2 \,.\, AB$.

$\therefore$ the depth of the centre of pressure is

$$\frac{\frac{3}{4}\,.\,\frac{1}{3}CD+\frac{1}{2}\,.\,\frac{1}{6}AB}{\frac{1}{3}CD+\frac{1}{6}AB}h=\frac{3CD+AB}{4CD+2AB}h.$$

$\therefore$ it divides the breadth in the ratio $\dfrac{3CD+AB}{CD+AB}$.

$\therefore$ it will be at the intersection of AC and BD if

$$\frac{3CD+AB}{CD+AB}=\frac{AB}{CD} \quad \text{or} \quad AB^2=3CD^2.$$

131. This problem should be erased.

132. If x is the height of the cylinder with which the water is in contact, and if A is the vertex of the paraboloid, and AN the depth of the vertex below the highest level of the water,

$$v=\pi r^2(x-AN)+\tfrac{1}{2}\pi r^2 \,.\, AN,$$

or

$$v=\pi r^2\left(x-\frac{r^2\omega^2}{4g}\right);$$

whole pressure

$$=g\rho \,.\, 2\pi rx \,.\, \frac{x}{2}$$

$$=g\rho\pi r\left(\frac{v}{\pi r^2}+\frac{r^2\omega^2}{4g}\right)^2,$$

and this is greatest when $5\pi\omega^2r^4=12gv$.

133. Let v be the volume of water displaced in the position of equilibrium, and $v+ky$ the volume of the other liquid displaced when there is equilibrium;

then $$w=1 \cdot v=s(v+ky).$$

Placing the hydrometer in the liquid so that v is displaced, and letting it go, the acceleration when it has descended through the space x

$$=\frac{w-s(v+kx)}{m} \propto y-x,$$

the motion is therefore the same as that of a particle attracted to a centre of force which varies in proportion to the distance.

Hence the hydrometer will descend to the depth $2y$,

i.e. $2w(1-s)/ks$.

134. Let r be the radius of the sphere, x the depth of its centre below the surface of the water.

The distance between the centre and the plane of contact, being the sub-normal, is $2a$.

Also $$r^2=4a(a+c).$$

The area of the circle of contact is $4\pi ac$.

The pressure on the sphere is equal to the weight of the water which would be contained in a cylinder whose base is the circle of contact, and whose height is $x+2a$, together with that contained by the segment of the sphere cut off by the plane of contact. The volume of this segment is

$$\tfrac{2}{3}\pi r^2(r-2a)-\tfrac{1}{3} \cdot 2a \cdot 4\pi ac,$$

$$\therefore \quad \tfrac{2}{3}\pi r^3-\tfrac{4}{3}\pi r^2 a-\tfrac{8}{3}\pi a^2 c+4\pi ac(x+2a)=\tfrac{2}{3}\pi r^3.$$

Using the above value of r^2, this gives

$$x=4a^2/3c.$$

135. Consider the water above a plane touching the *lowest point of* the sphere in the first case.

Its volume is $\pi R^2 . 2r - \frac{4}{3}\pi r^3 = 2\pi r\,[R^2 - \frac{2}{3}r^2]$.

Its C.G. is at a height r.

When the sphere is gone the height of the C.G. is

$$\frac{\pi r\,[R^2 - \frac{2}{3}r^2]}{\pi R^2},$$

$\therefore$ it has fallen a distance $\frac{2}{3}\frac{r^3}{R^2}$,

$\therefore$ the loss of potential energy

$$= \frac{W}{\frac{4}{3}\pi r^3} . 2\pi r \left[R^2 - \frac{2}{3}r^2\right] \times \frac{2r^3}{3R^2}$$
$$= Wr\,[3R^2 - 2r^2] \div 3R^2.$$

When the sphere is half out of the water, the height of the water is

$$\frac{2r}{3}\,\frac{3R^2 - r^2}{R^2}.$$

The height of its C.G. is

$$\frac{\frac{2\pi r}{3}(3R^2 - r^2) . \frac{r}{3}\frac{3R^2 - r^2}{R^2} - \frac{2}{3}\pi r^3 . \left(\frac{2r}{3} . \frac{3R^2 - r^2}{R^2} - \frac{3r}{8}\right)}{\frac{2\pi r}{3}(3R^2 - 2r^2)},$$

$\therefore$ it has fallen $\frac{r^3}{8R^2}\,\frac{13R^2 - 8r^2}{3R^2 - 2r^2}$.

Loss of potential energy

$$= \frac{W}{2R^2}\,\frac{r^2}{8R^2}(13R^2 - 8r^2) = \frac{39R^2 - 24r^2}{48R^2 - 32r^2} \times \text{former loss.}$$

When the sphere leaves the water, w being its weight, it has gained in potential energy

$$w . \frac{2}{3}\frac{r^3}{R^2},$$

$\therefore$ its K. E. must be

$$\frac{Wr}{3R^2}(3R^2 - 2r^2) - w . \frac{2r^3}{3R^2}.$$

Its velocity is therefore

$$\sqrt{\frac{gr}{3R^2}\left(\frac{W}{w}(3R^2-2r^2)-2r^2\right)}.$$

136. Let $a\cos\alpha$ be the height of the free surface of the fluid at rest above the surface, a being the radius.

The volume of the rest of the sphere is

$$\frac{\pi a^3}{3}(1-\cos\alpha)^2(2+\cos\alpha).$$

If the rotating liquid rise to a height $a\cos\theta$ above the centre, the free space is

$$\frac{\pi a^3}{3}(1-\cos\theta)^2(2+\cos\theta)+\pi a^2\sin^2\theta\,.\,\frac{\omega^2 a^2\sin^2\theta}{4g},$$

$$\therefore\quad \cos^3\alpha-3\cos\alpha-\cos^3\theta+3\cos\theta=\frac{3\omega^2 a}{4g}\sin^4\theta.$$

The greatest elevation is

$$a(\cos\theta-\cos\alpha).$$

The greatest depression is

$$\frac{\omega^2 a^2\sin^2\theta}{2g}-a(\cos\theta-\cos\alpha).$$

The latter is greater than the former if

$$\frac{\omega^2 a}{4g}\sin^2\theta>\cos\theta-\cos\alpha.$$

Now $$\frac{\omega^2 a}{4g}\sin^2\theta/(\cos\theta-\cos\alpha)$$

$$=\frac{1}{3}\,\frac{3-(\cos^2\theta+\cos\alpha\cos\theta+\cos^2\alpha)}{\sin^2\theta}.$$

$\therefore$ the above inequality holds if

$$2\cos^2\theta-\cos^2\alpha-\cos\alpha\cos\theta \text{ is positive.}$$

But $$\cos\theta>\cos\alpha.$$

$\therefore$ this is true.

$\therefore$ the greatest depression exceeds the greatest elevation.

137. If water falls from a height h its velocity

$$v=\sqrt{2gh}.$$

If κ is the section of the tube, the volume of water which has its momentum destroyed in the time t is $v\kappa t$, and if m is the mass of unit volume, the momentum destroyed per unit of time $= mv^2\kappa$.

This is the pressure exerted by the falling water, and therefore if it support a column of the height x,

$$mv^2\kappa = m\kappa xg, \quad \text{and} \quad \therefore \quad x=2h.$$

138. Erase the centre of repulsive force.

The attraction at any point distant r from the centre is proportional to r, by Leibnitz's theorem.

Take this to be μr, and let $g=\mu c$.

Then if from O the centre of the sphere we measure OC downwards so that $OC=c$, the surfaces of equal pressure are spheres having their centre at C.

139. If a be the area of the jet, v the velocity of efflux, p the pressure, x the depth.

In a short time τ a mass $\rho av\tau$ emerges with velocity v.

The work done is $p\,.\,a\,.\,v\tau$.

$$\therefore \quad pa\,.\,v\tau = \tfrac{1}{2}\rho av^3\tau,$$

$$\therefore \quad v^2=\frac{2p}{\rho}=2gx.$$

The mass ρkvt emerging

$$=\rho kt\sqrt{\frac{2p}{\rho}}=kt\sqrt{2p\rho}.$$

The momentum emerging ρkv^2t

$$=2pkt.$$

140. If f is the acceleration, downwards, of the bucket

$$f=\frac{mg}{2M+m}.$$

Let v be the volume of the cork, σ its density, so that $m=\sigma v$, and T the tension of the string.

If P is the upward pressure of the water, taking unity as the density of water,

$$vf=vg-P, \therefore P=\frac{2Mvg}{2M+m}.$$

Now $$mf=mg-P+T,$$

$$\therefore \quad T=\frac{2Mmg}{2M+m}\left(\frac{1}{\sigma}-1\right).$$

If V is the volume of water in the bucket, and h its height, $$V=\pi r^2 h.$$

Pressure on curved surface at first

$$=g\rho\pi r h^2=g\rho\frac{V^2}{\pi r^3}.$$

Afterwards, pressure $=(g-f)\,\rho\,\dfrac{(V+v)^2}{\pi r^3}$,

and this is greater or less than before according as

$$\frac{2M}{2M+m}(V+v)^2 > \text{ or } < V^2,$$

or $$\frac{v}{V} > \text{ or } < \sqrt{1+\frac{m}{2M}}-1.$$

ADDENDA.

1. In the solution of 16, page 29, it is intended that the second paragraph should apply to the cases of both the hole in the lid of a teapot, and of a vent-peg. In each case the liquid flows out more freely when its upper surface is exposed to the external atmosphere, and, in fact, it sometimes happens that, without opening a vent-peg, the liquid from a cask will not flow out at all.

The first paragraph should be erased.

2. The problem, 83, of Miscellaneous Problems, can be solved without assuming the property of the maximum area.

The fluid pressure being normal to the surfaces, and the same at all points of the same horizontal plane, the problem at once resolves itself into the equilibrium of a polygon of jointed rods, in one plane, the rods being acted upon, outwards, by normal forces at their middle points, proportional to their lengths.

Consider the equilibrium of four of the rods forming the polygon $A_1A_2A_3A_4$. Taking any one rod A_1A_2, the resultant of the stresses at A_1 and A_2 must bisect the rod at right angles.

These stresses are therefore equal, and consequently it follows that the stresses at all the joints are the same.

Let θ be the inclination to A_2A_1 of the stress at A_2 or A_1, ϕ the inclination to A_2A_3 of the stress on A_2 or A_3, and ψ the inclination to A_3 of the stress at A_3 or A_4.

Then, pr being the force on a rod of length r, and R the stress at each joint,

$$p.A_1A_2=2R\sin\theta,\quad p.A_2A_3=2R\sin\phi,\quad p.A_3A_4=2R\sin\psi.$$

$$\therefore\quad \frac{A_1A_2}{\sin\theta}=\frac{A_2A_3}{\sin\phi}=\frac{A_3A_4}{\sin\psi}.$$

Let the straight line through A_2 at right angles to the direction of R intersect in O the straight line bisecting A_1A_2 at right angles, and in O' the straight line bisecting A_2A_3 at right angles.

Then

$$A_2O=\tfrac{1}{2}A_1A_2\operatorname{cosec}\theta=\tfrac{1}{2}A_2A_3\operatorname{cosec}\phi=A_2O',$$

and therefore O and O' are coincident, and O is the centre of the circle passing through $A_1A_2A_3$.

Moreover, OA_2 and OA_3 are perpendicular to the directions of R at A_2 and A_3, so that these directions are tangents to the circle.

Again it can be shewn, in exactly the same manner, that the straight line bisecting A_3A_4 at right angles passes through the point O.

Hence it follows that A_1, A_2, A_3, A_4 are concyclic, and therefore that all the angular points of the polygon are concyclic.

Finally, each of the expressions $c_1\operatorname{cosec}a_1$, $c_2\operatorname{cosec}a_2$, &c., represents the diameter of the circle.

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